Math 43: Spring 2020
Lecture 3 Part 3

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Complex-Valued Functions

Our goal is to study functions

$$f : D \subset \mathbf{C} \to \mathbf{C}.$$

Almost always, D is going to be a domain. Note that we could just as easily view f as a function

$$f : D \subset \mathbf{R}^2 \to \mathbf{R}^2.$$

In multivariable calculus we called such things vector fields. In particular,

$$f(x, y) = \big(u(x, y), v(x, y)\big)$$

where $u, v : D \subset \mathbf{R}^2 \to \mathbf{R}$ are real-valued functions of two variables. In our new complex world, we will write

$$f(x + iy) = u(x, y) + iv(x, y),$$

and both u and v are old friends.

This is Easier that it Sounds

Example

Let $f(z) = \frac{1}{1+z}$. Find u and v such that $f(x+iy) = u(x,y) + iv(x,y)$.

Solution.

$$\begin{aligned} f(z) &= \frac{1}{1+z} \cdot \frac{1+\overline{z}}{1+\overline{z}} = \frac{1+\overline{z}}{|1+z|^2} \\ &= \frac{1+x-iy}{(x+1)^2+y^2} = \frac{1+x}{(x+1)^2+y^2} - i\frac{y}{(x+1)^2+y^2}. \end{aligned}$$

Therefore

$$u(x,y) = \frac{1+x}{(x+1)^2+y^2} \quad \text{and} \quad v(x,y) = \frac{-y}{(x+1)^2+y^2}.$$

□

Visualizing Functions

Remark

We don't have enough dimensions to draw graphs of functions $f : C \subset \mathbf{C} \to \mathbf{C}$. One—admittedly rather poor—substitute is to view f as a transformation from one copy of $\mathbf{C}$ to another.