Math 43: Spring 2020
Lecture 8 Part 2

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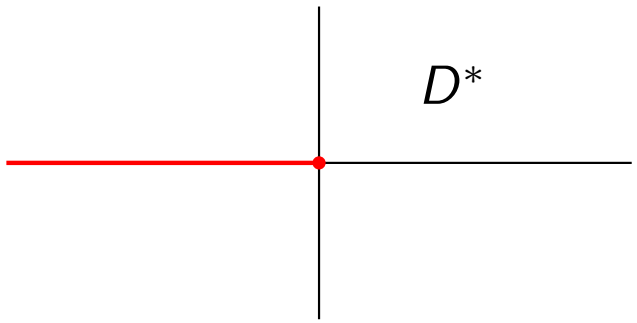
The Logarithm as a Analytic Function

Remark

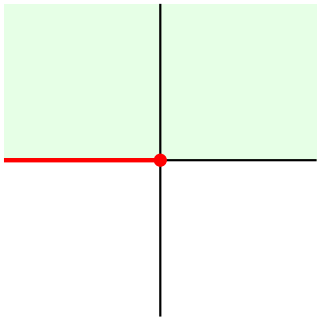
Now that we have a complex logarithm, we want to see if we can add it our collection of analytic functions. The first problem is that $\log(z)$ is set valued and only defined on the punctured plane $\mathbf{C} \setminus \{0\}$. Hence we will have to introduce a single-valued version and work with it on a restricted domain. Of course, the natural first candidate is the principal branch $f(z) = \operatorname{Log}(z) = \ln\big(|z|\big) + i \operatorname{Arg}(z)$. But $\operatorname{Arg}(z)$ has a nasty jump discontinuity along the whole negative real-axis! Hence it is natural to consider the domain $D^* = \mathbf{C} \setminus (-\infty, 0]$. We will adopt this notation for the future.



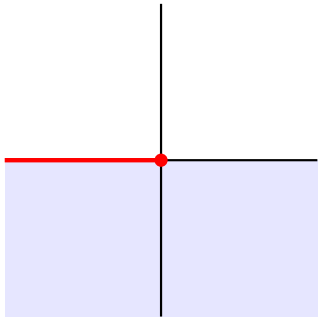
Local Properties

Remark

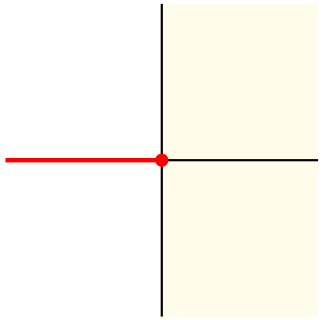
Properties of a function $f : D \subset \mathbf{C} \to \mathbf{C}$ like continuity and differentiablity at a point z_0 are what we often call local properties. By this, we just mean that they are determined only by the values of f in a neighborhood of z_0. This means that if we can write D as a union of open sets on which f is, say, continuous, the f is continuous on all of D. We will do this in the next result by writing $D^* = \mathbf{C} \setminus (-\infty, 0]$ as the union of the open upper half-plane, the open lower half-plane, and the open right half-plane.



(a) $\mathrm{Im}(z) > 0$



(b) $Im(z) < 0$



(c) $\mathrm{Re}(z) > 0$

$\mathrm{Arg}(z)$ is continuous

Lemma

$f(z) = \mathrm{Arg}(z)$ is continuous in D^*.

Proof.

If $y > 0$, then $f(x+iy) = \cos^{-1}\left(\frac{x}{r}\right) = \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$. Hence f is continuous in the open upper half-plane. However, if $y < 0$, then $f(x+iy) = -\cos^{-1}\left(\frac{x}{r}\right)$ and f is continuous in the open lower half-plane. But if $x > 0$, then $f(x+iy) = \tan^{-1}\left(\frac{y}{x}\right)$. Hence f is continuous in the open right half-plane. Since continuity is a local property, f is continuous in all of D^*. $\square$

Corollary

$g(z) = \mathrm{Log}(z)$ is continuous on D^*.

Proof.

We already know $z \mapsto \ln\big(|z|\big)$ is continuous on D^*. Thus, using the lemma, we know both the real and imaginary parts of g are continuous. $\square$

Enough with the Warm-Up Act

Theorem

The function $g(z) = \mathrm{Log}(z)$ *is analytic in* D^* *and* $g'(z) = \dfrac{d}{dz}\big(\mathrm{Log}(z)\big) = \dfrac{1}{z}$.

Proof.

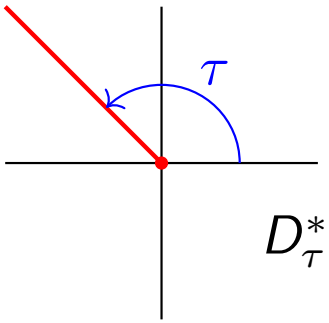
Fix $z_0 \in D^*$. Write $w = \mathrm{Log}(z)$ and $w_0 = \mathrm{Log}(z_0)$. Hence $z = e^w$ and $z_0 = e^{w_0}$. Thus $\frac{g(z)-g(z_0)}{z-z_0} = \frac{w-w_0}{e^w-e^{w_0}}$. Furthermore, if $z \neq z_0$, then $z = e^w$ and $z_0 = e^{w_0}$ forces $w \neq w_0$. Then $g'(z_0) = \lim\limits_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0} = \lim\limits_{w\to w_0} \frac{w-w_0}{e^w-e^{w_0}}$. But $\lim\limits_{w\to w_0} \frac{w-w_0}{e^w-e^{w_0}} = \lim\limits_{w\to w_0} \frac{1}{\frac{e^w-e^{w_0}}{w-w_0}} = \frac{1}{e^{w_0}} = \frac{1}{z_0}$. $\square$

Nothing Special about $\mathrm{Log}(z)$

There is nothing sacred about the principal branch, $\mathrm{Log}(z)$, of $\log(z)$. We just acted crudely to produce a single-valued version of $\log(z)$ by removing the ray $\arg(z) = \pi$. We could pick any other ray and produce just as good—or bad—a function. For example, let $\tau \in \mathbf{R}$ and define

$$\mathcal{L}_\tau(z) = \ln\big(|z|\big) + i \arg_\tau(z).$$

(Recall that $\arg_\tau(z) \in \arg(z) \cap (\tau, \tau + 2\pi]$.) Since $\arg_\tau(z)$ has a jump discontinuity along the ray $\arg(z) = \tau$, we can repeat the above proof to show that $\mathcal{L}_\tau(z)$ is analytic in $D^*_\tau = \mathbf{C} \setminus \{\, re^{i\tau} : 0 \le r < \infty \,\}$. Moreover $\frac{d}{dz}\big(\mathcal{L}_\tau(z)\big) = \frac{1}{z}$.



As Promised

Corollary

The function $u(z) = \ln\big(|z|\big)$ *is harmonic in the punctured plane* $\mathbf{C} \setminus \{0\}$. *Furthermore, the functions* $v_\tau(z) = \arg_\tau(z)$ *are harmonic in* D^*_τ. *(Note that* $\operatorname{Arg}(z) = \arg_{-\pi}(z)$*.)*

Proof.

Since we can write out specific formulas for $u(z)$ and $\arg_\tau(z)$, we know that they have continuous second partials as functions of (x, y). Since $\mathcal{L}_\tau(z) = u(z) + i \arg_\tau(z)$ is analytic in D^*_τ, we see that u and $\arg_\tau$ must be harmonic in D^*_τ as the real and imaginary parts, respectively, of an analytic function. But then u must be harmonic in $D^* \cup D^*_0 = D^*_{-\pi} \cup D^*_0$. $\square$

Let's take another break.