Math 46: Applied Math: Midterm 1

2 hours, 50 points total, 6 questions worth varying number of points

1. [7 points]

   In 1940 the Russian applied mathematician A. Kolmogorov assumed there was a law for turbulent fluid flow relating the four quantities: l (length), E (energy, units of ML^2T^{-2}), ρ (density, mass per unit volume), and R (dissipation rate, energy per unit time per unit volume). Using this assumption and the Buckingham Pi Theorem, state the simple form the law must have. Show that there is a (famous!) scaling relation $E = \text{const} \cdot l^\alpha$ when other parameters are held constant; give α.

2. [16 points. Note part c, worth 7 points, is independent of the others]
   A nonlinear damped oscillator is given by the initial-value problem

$$my'' + ay' + ky^3 = 0 \qquad y(0) = 0 \qquad my'(0) = I$$

   (a) If m is a mass, find the dimensions of the other three parameters a, k, I (recall y is a displacement, *i.e.* length).

   (b) Write down *two* length scales and *two* time scales.

   (c) Show that when the model is non-dimensionalized using scaling appropriate for the *small mass* limit (choose time and length scales which don't involve m), the IVP

$$\varepsilon y'' + y' + y^3 = 0 \qquad y(0) = 0 \qquad \varepsilon y'(0) = 1$$

   results. What is ε in terms of the original parameters?

(d) Find a *leading-order* perturbation approximation to the solution of the IVP from (b), and give a crude sketch showing any key features. Here it is written out again:

$$\varepsilon y'' + y' + y^3 = 0 \qquad y(0) = 0 \qquad \varepsilon y'(0) = 1 \qquad \varepsilon \ll 1$$

3. [6 points] Find a 3-term perturbation approximation to the solution of the IVP

$$y' = \frac{1}{1 + \varepsilon y^2 y'} \qquad\qquad y(0) = 0$$

4. [5 points] Find the leading-order perturbation approximation to all roots of $\varepsilon x^3 - x - 2 = 0$ for $\varepsilon \ll 1$.

5. [5 points] Find the WKB approximation for the large eigenvalues λ of

$$y'' + 4\lambda e^x y = 0 \qquad\qquad y(0) = y(1) = 0$$

6. [12 points] Short answers:

(a) As $\varepsilon \to 0$ does the function $f(x, \varepsilon) = \varepsilon \tan(x)$ converge uniformly to zero on $(-\pi/4, \pi/4)$? On $(0, \pi/2)$? (Why?)

(b) What can you say about stability and local asymptotic stability for the system $x' = -2x + y$, $y' = 4x + y$?

(c) A linearization of a nonlinear system of two coupled ODEs at a critical points gives the Jacobean matrix $\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$. What can you conclude about stability?

(d) At which end(s) would you expect the BVP $\varepsilon y'' + (1/2 - x)y' + y = 0$ with $y(0) = a$ and $y(1) = b$ to be able to support a boundary layer for $\varepsilon \ll 1$? [Do not solve the whole thing!]

(e) State briefly in what class of problem the Poincaré-Linstedt method is needed, and what problem it fixes.

(f) Sketch orbits in the (x, x') plane for a particle at location $x(t)$ subjected to a force $F(x) = x^2 - 1$. BONUS: What kinds of motion are possible and in what energy range?

Useful formulae:

WKB approximation

$$y = \frac{1}{\sqrt{k(x)}} e^{\pm \frac{1}{\varepsilon} \int k(x) dx}$$

Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$$