Math 46, Applied Math (Spring 2009): Midterm 1

2 hours, 50 points total, 6 questions worth varying number of points

1. [9 points] A gas bubble of mean radius R, containing gas at mean pressure P, in a surrounding fluid of density ρ, can oscillate at frequency ω. This is actually useful in enhancing ultrasound reflection in medical imaging. [Hint: pressure is force per unit area; force is mass times acceleration]

   (a) How many (independent) dimensionless quantities are there? Give them. [Hint: a dimensions matrix will help].

   (b) If a physical law relates the four parameters given in the problem, how must ω scale with R when the other two parameters are fixed?

   (c) If instead a physical law relates the above four to a fifth parameter v, the velocity of the bubble moving through the fluid, use the Buckingham Pi Theorem to deduce whether the scaling between ω and R you found above *must* still hold when the other three parameters are fixed. (Explain)

2. [7 points] Consider the algebraic equation

$$\varepsilon x^3 + x + 1 = 0$$

(a) Find leading-order approximations to all solutions valid for small $\varepsilon \ll 1$

(b) Find a 2-term approximation to the root which is finite as $\varepsilon \to 0$

(c) [BONUS] Answer again (a) above if the equation is changed to $\varepsilon^2 x^4 + \varepsilon x^3 + x + 1 = 0$

3. [9 points] Use singular perturbation methods to find a uniform approximate solution to the boundary-value problem

$$\varepsilon y'' - 2y' - e^y = 0, \qquad \varepsilon \ll 1, \qquad y(0) = 0, \qquad y(1) = 0$$

As always, remember to check and explain the location of any boundary layer(s). [Hint: if you can't solve an ODE, express things in terms of its limiting value(s)]

4. [10 points] Short answer questions.

(a) Find a WKB approximation to the nth eigenvalue of $\varepsilon^2 y'' + (1+x)^2 y = 0$ with $y(0) = y(1) = 0$ for large n.

(b) Sketch a bifurcation diagram showing equilibria and stability for the ODE $du/dt = u^2 - h^2$, as the parameter h varies.

(c) Prove or disprove the following claim: $\dfrac{1}{\log \varepsilon} = o(\varepsilon)$ as $\varepsilon \to 0^+$

(d) Is $f(t, \varepsilon) = \sin(\varepsilon t)$ pointwise, and/or uniformly, convergent to zero on the interval $t \in (0, +\infty)$, as $\varepsilon \to 0$? (briefly explain)

(e) [BONUS] Is it possible for a function $f(t, \varepsilon)$ to be uniformly convergent on some interval of t, but not pointwise convergent, as $\varepsilon \to 0$? (Give an example or explain.)

5. [9 points] Consider the perturbed initial-value problem for $y(t)$ on $t > 0$,

$$y'' + y = 4\varepsilon y(1 - y'^2), \qquad 0 < \varepsilon \ll 1, \qquad y(0) = 1, \qquad y'(0) = 0$$

(a) Use the Poincaré-Lindstedt method to give a 2-term uniform approximation. [Hint: set $\tau = \omega t$ where ω is perturbed from the value 1. Don't forget to match initial conditions.]

(b) Has switching on the perturbation increased or decreased the period of the oscillator?

6. [6 points] Radioactivity is modeled by quantum particles leaking through a barrier according to the Schrödinger equation

$$y'' - \frac{\lambda}{x^{3/2}} y = 0, \qquad \text{for } x > 1,$$

where $\lambda \gg 1$ is a large positive parameter.

(a) Is the ODE oscillatory or growing/decaying?

(b) Write down a WKB approximation to the *general* solution

(c) Find a WKB approximation to y in the barrier region $x > 1$, if the initial value is $y(1) = 1$, and a condition $\lim_{x \to +\infty} y(x) = 0$ is imposed.

Useful formulae:

non-oscillatory WKB approximation

$$y = \frac{1}{\sqrt{k(x)}} e^{\pm\frac{1}{\varepsilon}\int k(x)dx}$$

Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Error function [note $\mathrm{erf}(0) = 0$ and $\lim_{z\to\infty}\mathrm{erf}(z) = 1$]:

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2} ds$$

Euler relations

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Power-reduction identities

$$\begin{aligned} \cos^3\theta &= \frac{1}{4}(3\cos\theta + \cos 3\theta) \\ \cos^2\theta\sin\theta &= \frac{1}{4}(\sin\theta + \sin 3\theta) \\ \cos\theta\sin^2\theta &= \frac{1}{4}(\cos\theta - \cos 3\theta) \\ \sin^3\theta &= \frac{1}{4}(3\sin\theta - \sin 3\theta) \end{aligned}$$