MATH 46 WORKSHEET : convergence of funcs. Barnett 4/20/08

A) Is the sequence $f_n(x) = \begin{cases} 1 & x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$ convergent to 0, on $(0,1)$

in the sense of $\begin{cases} \text{pointwise} & ? \\ \text{uniform} & ? \\ L^2 & ? \end{cases}$ (careful: interval is $(0,1)$ not $[0,1]$)

B) Same for sequence $f_n(x) = \begin{cases} \sqrt{n} & x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$ on $(0,1)$.

$\begin{cases} \text{pointwise} ? \\ \text{uniform} ? \\ L^2 ? \end{cases}$

C) Now consider on <u>unbounded</u> interval $(-\infty, \infty)$, $f_n(x) = \begin{cases} \frac{1}{n} & |x| < n \\ 0 & \text{otherwise} \end{cases}$

$\begin{cases} \text{pointwise} ? \\ \text{uniform} ? \\ L^2 ? \end{cases}$

D) Modify example C) so that convergence is pointwise & uniform but <u>not</u> L^2:

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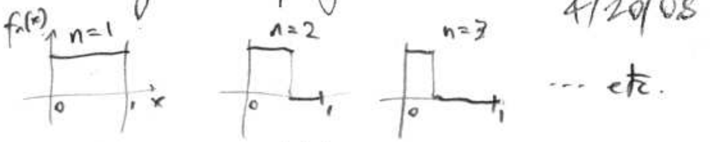
~ SOLUTIONS ~



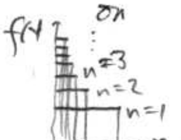
A) Is the sequence $f_n(x) = \begin{cases} 1 & x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$ } sketch. convergent to 0, on (0,1)

in the sense of

- pointwise ? yes. (for any $x \in (0,1)$, $\frac{1}{n}$ will eventually be smaller, so all f's zero). (careful: interval is (0,1) not [0,1]) it's not pointwise conv. on [0,1]!
- uniform ? no : $\max_{x\in(0,1)} |f_n(x)| = 1 \ \forall n, \not\to 0$
- L^2 ? yes: $\|f_n\|^2 = \int_0^1 f_n(x)^2\,dx = \frac{1}{n} \to 0$ as $n \to \infty$

Note: $n\to\infty$ plays the role of $\varepsilon \to 0$ in earlier presentation of pointwise vs. uniform conv.

B) Same for sequence $f_n(x) = \begin{cases} \sqrt{n} & x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$ on (0,1).



- pointwise ? yes, same reason as above
- uniform ? no, since $\max_{x\in(0,1)} |f_n(x)| = \sqrt{n} \not\to 0$.
- L^2 ? no, since $\|f_n\|^2 = \int_0^1 f_n^2(x)\,dx = \int_0^{1/n} \sqrt{n}^2\,dx = 1 \not\to 0$.

C) Now consider on unbounded interval $(-\infty,\infty)$, $f_n(x) = \begin{cases} \frac{1}{n} & |x| < n \\ 0 & \text{otherwise} \end{cases}$

- pointwise ? yes : for each $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = 0$.
- uniform ? yes : $\max_{x\in\mathbb{R}} |f_n(x)| = \frac{1}{n} \longrightarrow 0$. as $n\to\infty$.
- L^2 ? yes: $\|f_n\|^2 = \int_{-n}^{n} \left(\frac{1}{n}\right)^2 dx = \frac{2}{n} \to 0$ as $n \to \infty$

D) Modify example C) so that convergence is pointwise & uniform but not L^2

Change height so vanishes more slowly: $f_n(x) = \begin{cases} \frac{1}{n^\alpha} & |x| < n \\ 0 & \text{otherwise} \end{cases}$

for any $\alpha \le \frac{1}{2}$. then $\|f_n\|^2 = \int_{-n}^{n} \frac{1}{n^{2\alpha}}\,dx = 2n^{1-2\alpha} \not\to 0$

Note: unif $\Rightarrow$ pointwise always (ie unif. is stronger statement); Unif $\Rightarrow L^2$ on bounded interval ; on unbounded interval unif & L^2 are indep.