

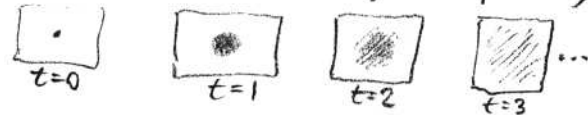
SOLUTIONS

Here we explore the fundamental soln for the heat equation - without calculus!

Pulse of energy sized e released at origin at time $t=0$. The medium has heat capacity c (energy per volume per degree) and thermal conductivity \mathcal{K} (power per length per degree).

The temperature at distance r and time t is u (we take $u=0$ everywhere for $t < 0$)

$$E \begin{bmatrix} e & r & t & u & c & \mathcal{K} \\ 1 & & & & 1 & 1 \\ & 1 & & & -3 & -1 \\ & & 1 & & & -1 \\ & & & 1 & -1 & -1 \end{bmatrix}$$



a) Using fundamental units energy (E), length (L), time (T), temperature (Θ), fill in the dimensions of the $n=6$ quantities in the problem above (check with me).

b) Find $p=2$ independent dimensionless quantities. Since there's freedom, choose π_1 to not involve u : $\pi_1 = \dots \frac{cr^2}{\mathcal{K}t}$... note it's $\frac{\text{dist}^2}{\text{time}}$
 π_2 to not involve r : $\pi_2 = \dots \frac{ce^2}{\mathcal{K}^3 t^3 u^2}$ or $\frac{(\mathcal{K}t)^{3/2} u}{c^{1/2} e}$ if choose power of u to be 1

c) Buckingham Pi Theorem tells us $F(\pi_1, \pi_2) = 0$, so $\pi_2 = g(\pi_1)$
 From this derive a solution of the form $u = \left(\frac{ce^{1/2}}{(\mathcal{K}t)^{3/2}}\right) \tilde{g}\left(\frac{cr^2}{\mathcal{K}t}\right)$
 just be rearranging $\pi_2 = g(\pi_1)$ ↪ note not the same g as before (but unknown), so not important

d) if $r=0$ how must u scale with t ? (you may assume g has limits at 0, ∞ ; in fact PDEs tell us g is a gaussian!)
 $\tilde{g}(0)$ or $\tilde{g}(\infty)$ is some const.

$\Rightarrow u = (\text{const}) \frac{ce^{1/2}}{(\mathcal{K}t)^{3/2}} \sim t^{-3/2}$ if all else held const. decays

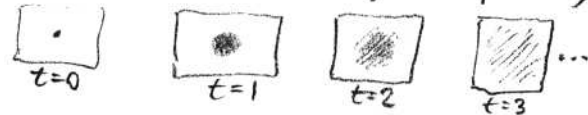
e) How does scaling in d) change in general space dimension d ? (we had $d=3$ above; note that \mathcal{K} has units $ET^{-1}L^{2-d}\Theta^{-1}$ in general d) Turns out $u \sim t^{-d/2}$
 Note $g(\pi_1) = e^{-\pi_1/2}$ a result we'll get to later. but have to redo lin. algebra.

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