MATH 46 WORKSHEET : Using WKB approx. 4/18/07 Barnett

Consider $\varepsilon^2 y'' - xy = 0 \qquad \varepsilon \ll 1$

For what domains is it oscillating?

evanescent (growing/decaying)?

Let's take $x > 1$ Write down $k(x)$:

Write general WKB solution:

Find coefficients so that BCs $y(1) = 1$, $\lim_{x\to\infty} y(x) = 0$ obeyed:

Rewrite the WKB solution so $x=1$ is lower limit of action integral:

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~ SOLUTIONS ~

Consider $\varepsilon^2 y'' - xy = 0$ $\varepsilon \ll 1$

For what domains is it oscillatory? $x<0$

evanescent (growing/decaying)? $x>0$

Let's take $x>1$ Write down $k(x)$: $k(x) = \sqrt{x}$

so $\int k(x)\,dx = \frac{2}{3}x^{3/2}$

Write general WKB solution:

$$y(x) = c_1 \frac{1}{x^{1/4}} e^{\frac{1}{\varepsilon}\frac{2}{3}x^{3/2}} + c_2 \frac{1}{x^{1/4}} e^{-\frac{1}{\varepsilon}\frac{2}{3}x^{3/2}}$$

notice the integration consts absorbed into c_1, c_2.

Find coefficients so that BCs $y(1) = 1$, $\underbrace{\lim_{x\to\infty} y(x) = 0}$ obeyed:

means $c_1 = 0$ since $e^{+\frac{1}{\varepsilon}\cdots} \to +\infty$ and would not have well-behaved limit.

$$1 = \frac{c_2}{(1)^{1/4}} e^{-\frac{1}{\varepsilon}\frac{2}{3}(1)^{3/2}} \quad \text{so} \quad c_2 = e^{\frac{1}{\varepsilon}\frac{2}{3}}$$

Rewrite the WKB solution so $x=1$ is lower limit of action integral:
sub. for c_2:

$$y(x) = e^{\frac{1}{\varepsilon}\frac{2}{3}} \frac{1}{x^{1/4}} e^{-\frac{1}{\varepsilon}\frac{2}{3}x^{3/2}} = \frac{1}{x^{1/4}} e^{-\frac{1}{\varepsilon}\frac{2}{3}(x^{3/2}-1)}$$

$$= \frac{1}{x^{1/4}} e^{-\frac{1}{\varepsilon}\underbrace{\int_1^x \sqrt{s}\,ds}}$$

so this is action integral with lower limit at left end of interval, $x=1$.