Barnett 10/1/09

MATH 53 : proof of exponential smallness of itinerary subintervals.

→ needed to prove sens. dep. in
→ we'll do it again later in Ch. 3.

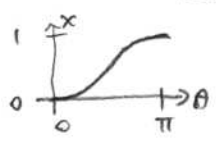
[This will also explain 1.15 from HW1 ... the trig one!]

It turns out if you map x to a new variable θ, then the logistic map $G(x) = 4x(1-x)$ becomes ridiculously simple: just doubling θ.

I.e:

$$x_n \xrightarrow{\quad G \quad} x_{n+1} = 4x_n(1-x_n)$$

$x = \frac{1-\cos\theta}{2}$ bijection



$$x = \frac{1-\cos\theta}{2}$$

$$\theta_n \longrightarrow \theta_{n+1} = 2\theta_n$$

← (called a "commutative diagram")

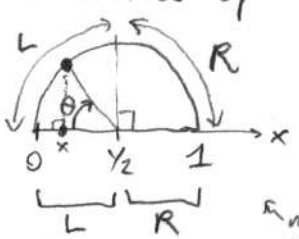
This states that iterating x_n to x_{n+1} via G is equivalent to converting to θ, doubling θ, then converting back again. (repeat this giving $\theta \to 2^k\theta$ for HW1 1.15)

Why does this work? If $x = \frac{1-\cos\theta}{2}$ then $\frac{1-\cos 2\theta}{2} = \sin^2\theta = 1-\cos^2\theta = 4\left(\frac{1-\cos\theta}{2}\right)\left(\frac{1+\cos\theta}{2}\right)$

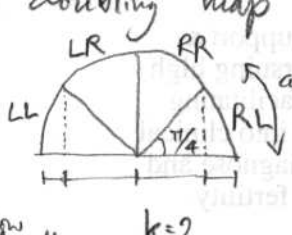
(doubling θ; by trig.; by diff of squares)

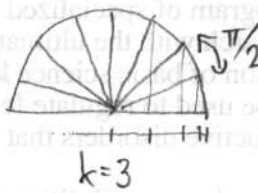
$$= 4x(1-x)$$

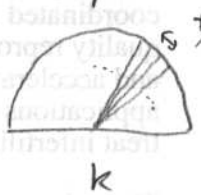
Itineraries of the doubling map $\theta \to 2\theta$ are really easy to compute:



↖ note how geometrically, $x = \frac{1-\cos\theta}{2}$







the subintervals are equally spaced in angle since doubling is simple

Projecting the semicircle down to $[0,1]$, we see that no subinterval of k symbols may exceed $\frac{1}{2} \cdot \frac{\pi}{2^k} = \frac{\pi}{2^{k+1}}$ in length along x.

(radius $\frac{1}{2}$, angle $\frac{\pi}{2^k}$)

QED.