Math 53: Chaos! 2015: Midterm 1

2 hours, 50 points total, 4 questions, points somewhat $\propto$ blank space. Good luck!

1. [11 points] Consider the map $f(\mathbf{x}) = \begin{bmatrix} 1/2 & 0 \\ -1 & -1/2 \end{bmatrix} \mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^2$.

   (a) Describe the object to which the unit disc maps under f. Include all relevant length(s) of the object. You do *not* have to give directions.

   (b) What is the area of the object you just described? [Reminder: unit disc has area π].

   (c) Do any points from inside the unit disc get mapped outside the unit disc by f?

   (d) What is the fate (asymptotic behavior) of orbits launched from all points $\mathbf{x} \in \mathbb{R}^2$, and why?

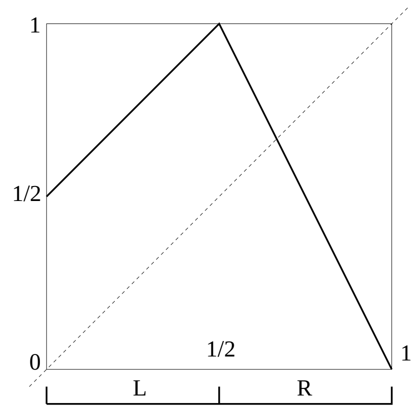
(e) Imagine the map is changed to $f\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x^2 + xy \\ y^3 \end{pmatrix}$. Classify the fixed point $(0, 0)$ as a sink/source/saddle.

2. [16 points]

Consider the function

$$f(x) = \begin{cases} x + 1/2, & x < 1/2 \\ 2(1 - x), & x \geq 1/2 \end{cases}$$

whose graph is shown to the right, mapping $[0, 1]$ to itself.



(a) Find and classify the fixed point(s).

(b) Find the Lyapunov exponent of the orbit starting at $x = 1/3$.

(c) Draw the *transition graph* for the intervals L and R shown.

(d) Give a specific x which is eventually periodic but not periodic.

(e) Prove that there exists orbits which are not fixed, periodic, or eventually periodic.

(f) To what subinterval does the subinterval LRR map under one application of f?

(g) Show the subintervals down to level 4, with their correct ordering. Take plenty of horizontal space.

BONUS Prove a positive lower bound on the Lyapunov exponent of all orbits which never hit 1/2.

3. [15 points] Consider the map $f(x) = 2x \pmod 1$ mapping the (periodic) interval $[0, 1)$ to itself.

   (a) Sketch a graph of f^2. How many fixed points of f^2 are there in $[0, 1)$?

   (b) Give the formula for the number of fixed points of f^k, for general $k \geq 1$.

   (c) Work out the first 4 rows of the "periodic table" for the map, which computes how many periodic orbits there are of periods 1 through 4:

(d) Give the mathematical definition of what it means for a given point x_0 to have *sensitive dependence.*

(e) Prove that each point $x_0 \in [0, 1)$ has sensitive dependence.

(f) Prove that each rational x_0 is eventually periodic.

4. [8 points] Unrelated short questions. Please explain only briefly.

(a) What precisely are plotted on the horizontal and vertical axes of a bifurcation diagram?

(b) Give the mathematical definition of a fixed point p being *sink*.

(c) What set comprises the unstable manifold for the saddle at $(0, 0)$ for the map $f(x, y) = (0.9x, -1.1y)$ in $\mathbb{R}^2$?

(d) Estimate how many iterations of the map $f(x) = 3x \pmod 1$ it takes for a computer rounding error of around 10^{-16} in the initial condition to cause an $O(1)$ change in the iterate. (You can leave an expression if you like.)