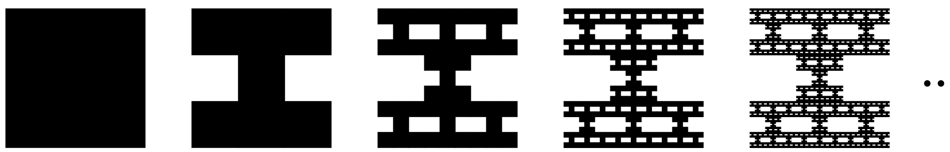
Math 53: Chaos! 2015: Midterm 2

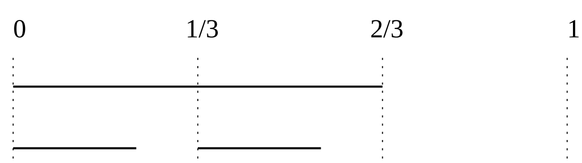
2 hours, 50 points total, 5 questions, points somewhat $\propto$ blank space. In each part be sure to explain your answer or give working. Good luck!

1. [5 points]

   (a) Compute the box-counting dimension of the limiting set formed by repeatedly treating each square as a 3×3 grid and removing the squares as shown:



2. [15 points] Consider the "last-third Cantor set" K defined as follows: i) Start with [0, 2/3]. ii) Split each remaining interval equally into two pieces then remove the last third of each piece. iii) Take the limit of repeating step ii) forever. For clarity, the results of the first two steps are shown here:



   (a) What numbers expressed in ternary (base 3) are in K?

(b) Is $x = 5/26$ in K?

(c) How many points are in K: finite, countably infinite, or uncountably infinite? Prove your answer.

(d) Is the set K dense in $[0,\ 2/3]$?

(e) Prove whether K has *zero measure* in $\mathbb{R}$ or not.

(f) Prove that there's an irrational in K. [Hint: construction]

(g) Describe a *probabilistic game* (set of maps involving a coin toss) that has K as its attractor.

3. [16 points] Consider a particle constrained to move in 1D in the potential $P(x) = x^2/2 - x^4/4$.

   (a) Write a coupled system of two first-order ODEs expressing the motion with no damping.

   (b) Sketch solutions in the phase plane $(x, \dot{x})$, showing the full variety of motions that can happen:

   (c) For what range of total energies can the particle have *periodic* motion? (Be precise whether each end of this interval is included.)

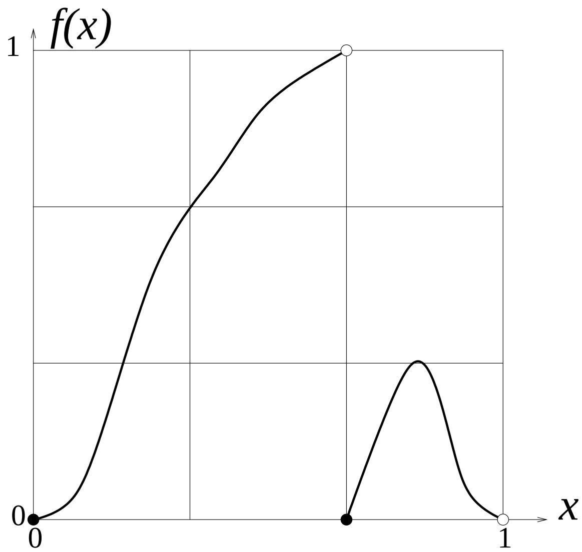
(d) What is the stability of the equilibrium point at $x = 1$? If it is possible to do so, prove your answer.

(e) Now assume a small amount of damping $c > 0$ is added. Sketch on a phase plane the *basin* of the equilibrium $x = 0$.

(f) For damping $c = 1$, state and, if possible, prove the stability of the equilibrium $x = 0$.

BONUS: What is the full range of periods possible?

4. [6 points] Consider the following graph of a map f, which can be viewed as continuous on the periodic interval $[0, 1)$.



(a) Draw the transition graph for f in the space above right, using the three intervals shown.

(b) List all periods of orbits that *must* exist, and prove your answer, mentioning what (if any) theorem(s) you used.

BONUS: Prove non-existence of any natural numbers missing from this list, taking care with endpoints.

5. [8 points] Unrelated short questions. Provide same amount of explanation as for other questions.

(a) What are the Lyapunov exponents of the cat torus map $\mathbf{x} = A\mathbf{x} \pmod 1$ where $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$?

(b) Is the number -2 in the Mandelbrot set?

(c) State the definition of an equilibrium point $\mathbf{v}$ of a flow $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ being (Lyapunov) *stable.*

(d) Argue whether the set of all periodic points of $G(x) = 4x(1-x)$ is finite, countably infinite, or uncountably infinite. [Hint: conjugacy]