MATH 53 WORKSHEET : Linear stability in 2D. 10/10/07 Barnett

a) Consider $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

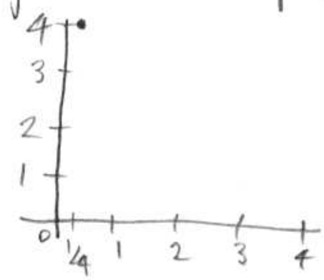
Find the condition on the eigenvalues such that $\vec{p} = \vec{0}$ is a...

sink :

source :

saddle :

b) For $A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ write and plot the first few iterates of $\vec{x}_0 = \left(\frac{1}{4}, 4\right)$



What curve do they all lie on?

c) Verify if $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ that $A^n = a^{n-1} \begin{pmatrix} a & n \\ 0 & a \end{pmatrix}$

Write out $A^n \vec{x}$ and use this to decide condition on a such that... sink

source.

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~ SOLUTIONS ~

a) Consider $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ a, b are eigenvalues

Find the condition on the eigenvalues such that $\vec{p} = \vec{0}$ is a...

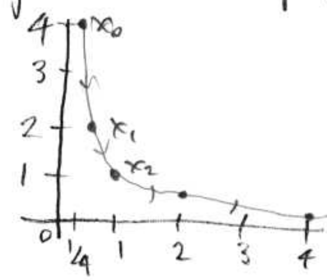
sink : $|a| < 1$ & $|b| < 1$

source : $|a| > 1$ & $|b| > 1$

saddle : $|a| < 1$ & $|b| > 1$ or $|a| > 1$ & $|b| < 1$

b) For $A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ write and plot the first few iterates of $\vec{x}_0 = (\frac{1}{4}, 4)$

$\begin{pmatrix} 1/4 \\ 4 \end{pmatrix} \to \begin{pmatrix} 1/2 \\ 2 \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 2 \\ 1/2 \end{pmatrix} \ldots$



What curve do they all lie on?

hyperbola $xy = 1$ since $x_0 y_0 = 1$ & $ab = 1$ ← determinant.

c) Verify if $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ that $A^n = a^{n-1} \begin{pmatrix} a & n \\ 0 & a \end{pmatrix}$

$n = 1$: $A^1 = a^0 \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ so formula holds for $n=1$.

Induction

$A^{n+1} = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} A^n = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} a^{n-1} \begin{pmatrix} a & n \\ 0 & a \end{pmatrix} = a^{n-1} \begin{pmatrix} a^2 & an + a \\ 0 & a^2 \end{pmatrix} = a^n \begin{pmatrix} a & n+1 \\ 0 & a \end{pmatrix} = A^{n+1}$ so formula correct.

Write out $A^n \vec{x}$ and use this to decide condition on a such that... sink $|a| < 1$ why?

$A^n \begin{pmatrix} x \\ y \end{pmatrix} = a^{n-1} \begin{pmatrix} ax + ny \\ ay \end{pmatrix}$

if $|a| > 1$ then $\lim_{n \to \infty} a^{n-1} \to \infty$ so source → source $|a| > 1$

If $|a| < 1$ then get $a^{n-1} ny$ as one term which might grow? Ratio test: $\lim_{n\to\infty} a^n n = 0$?

ratio $\frac{a^{n+1}(n+1)}{a^n n} = a \frac{n+1}{n} < 1$ eventually for all $n > N$, so limit is 0