MATH 53 WORKSHEET: Motion in a Potential

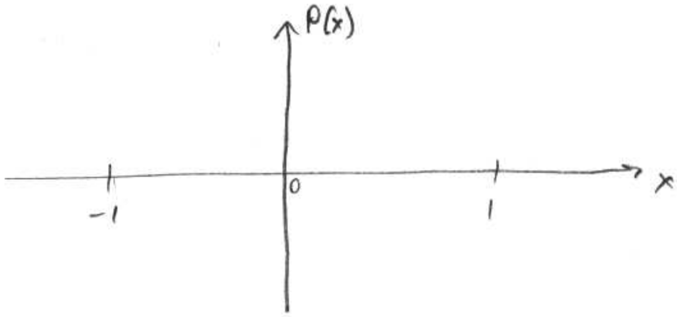
11/9/07
Barnett

Consider $x'' + 1 - 3x^2 = 0$

What is $\frac{dP}{dx} = ?$

$P(x) = ?$

Sketch

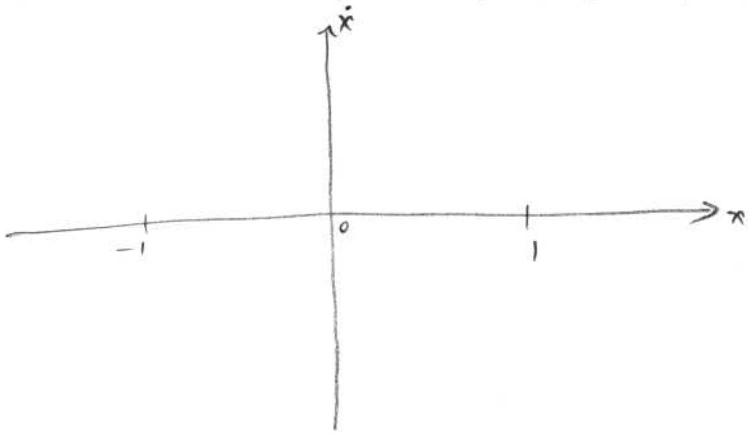


Write as 1st order system:

$x' =$

$y' =$

Sketch level curves of $E(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + P(x)$ in the phase plane:



Where are the equilibria?
(compute & show on plane).

What kinds of periodic orbits can happen? (What range of energies E may they have?)

When is the motion unbounded?

Deduce the stability using Jacobean $\vec{D}\vec{f}$ at the equilibria:

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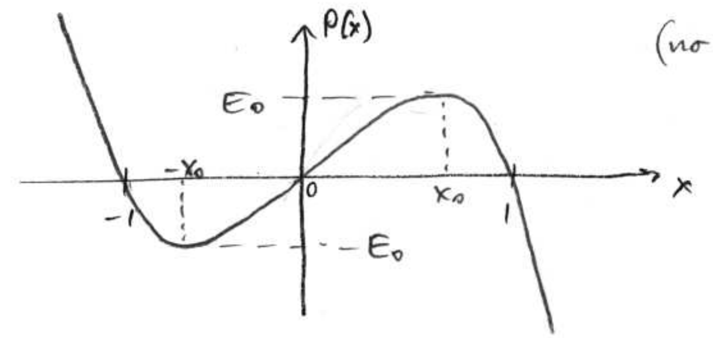
~ SOLUTIONS ~

Consider $x'' + \underbrace{\boxed{1 - 3x^2}}_{\text{this is } \frac{dP}{dx}} = 0$

(no minus sign)

Sketch



What is $\frac{dP}{dx} = ?$ $1 - 3x^2$

$P(x) = ?$ $x - x^3$

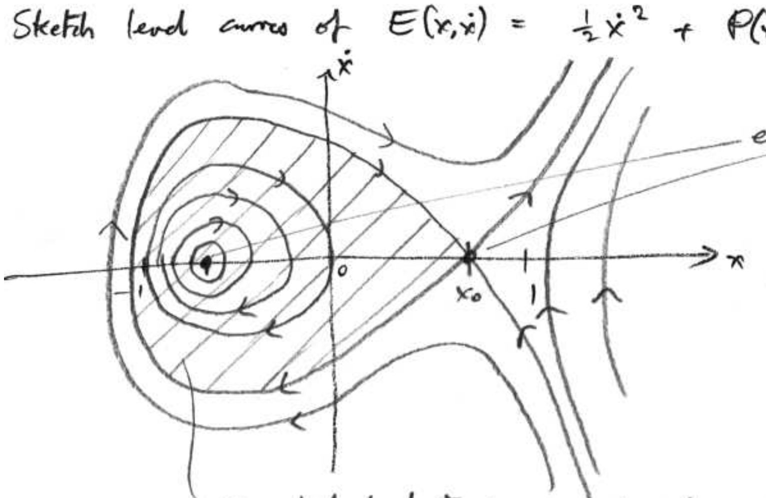
Write as 1st order system:

$x' = y$

$y' = -\frac{dP}{dx} = -1 + 3x^2$

ie $\vec{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -1 + 3x^2 \end{pmatrix}$

Sketch level curves of $E(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + P(x)$ in the phase plane:



region which leads to bounded periodic motion.

Where are the equilibria? (compute & show on plane).

$\frac{dP}{dx} = 0$ ie $1 - 3x^2 = 0$

$x_0 = \pm\frac{1}{\sqrt{3}}$

What kinds of periodic orbits can happen? (What range of energies E may they have?)

PO's oscillate either side of $-\frac{1}{\sqrt{3}}$ equilibrium.

they can only have $E < E_0 = P\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}}$

When is the motion unbounded?

It can be for any energy; but must be so for $E > E_0$.

Deduce the stability using Jacobean $\vec{D}\vec{f}$ at the equilibria:

$x_0 = \frac{1}{\sqrt{3}}$: $\vec{D}\vec{f} = \begin{pmatrix} 0 & 1 \\ 6x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2\sqrt{3} & 0 \end{pmatrix} \xrightarrow{\text{eigvals}} \lambda = \pm\sqrt{2\sqrt{3}}$ $\mathbb{C}$ saddle (unstable)

$x_0 = \frac{-1}{\sqrt{3}}$: $\vec{D}\vec{f} = \begin{pmatrix} 0 & 1 \\ -2\sqrt{3} & 0 \end{pmatrix}$ so $\lambda = \pm i\underbrace{\sqrt{2\sqrt{3}}}_{\text{angular freq at the bottom of well.}}$ $\mathbb{C}$ borderline case ($\mathrm{Re}\,\lambda = 0$), cannot deduce stability via the nonlin. stability thm.