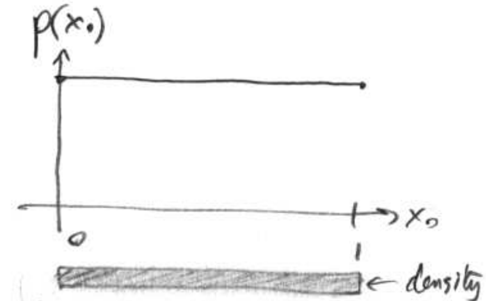
MATH 53 WORKSHEET : Fractals from probabilistic games

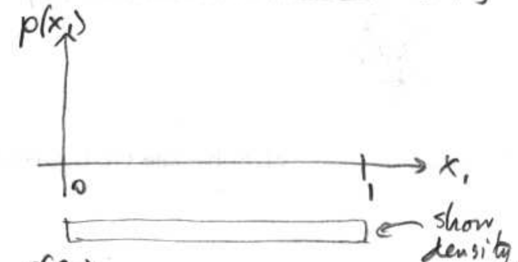
10/29/07
Barnett

- Apply $f_1(x) = \frac{x}{3}$ or $f_2(x) = \frac{x+2}{3}$ $\Big\}$ with equal probability of $1/2$ on each iteration.

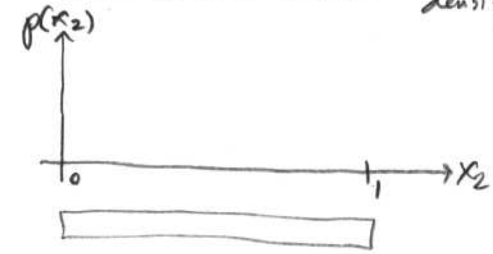
Starting with $p(x_0)$ uniform in $[0,1]$,



Find $p(x_1)$ and sketch
[Hint: what geometrically does f_2 do?]



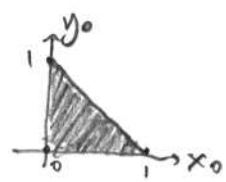
Find $p(x_2)$ and sketch



What is $p(x_n)$?
What is the limiting attractor set as $n \to \infty$?

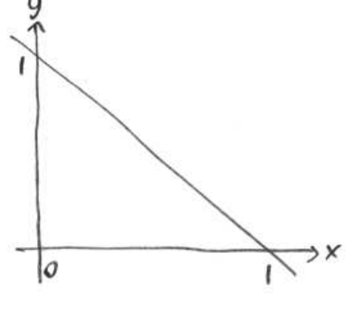
Prove an upper bound on the distance of x_n to this set [Hint: dist of $x_0 \le$?]

- Now try a 2D example : start with $\vec{x}_0$ uniform in triangle



$\vec{x}_0 = (x_0, y_0)$

Apply $\begin{cases} \vec{f}_1(\vec{x}) = \left(\frac{x}{2}, \frac{y}{2}\right) \\ \vec{f}_2(\vec{x}) = \left(\frac{x+1}{2}, \frac{y}{2}\right) \\ \vec{f}_3(\vec{x}) = \left(\frac{x}{2}, \frac{y+1}{2}\right) \end{cases}$ with equal probabilities $1/3$.



← Deduce the attractor set.

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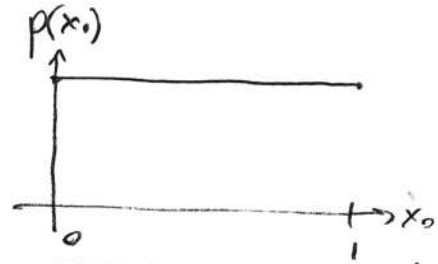
~~ SOLUTIONS ~~

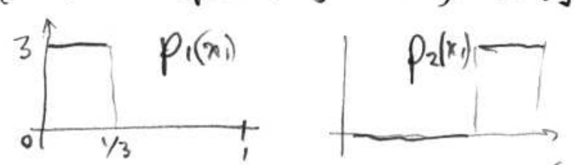
- Apply $\left.\begin{array}{l} f_1(x) = \frac{x}{3} \\ \text{or } f_2(x) = \frac{x+2}{3} \end{array}\right\}$ with equal probability of ½ on each iteration.

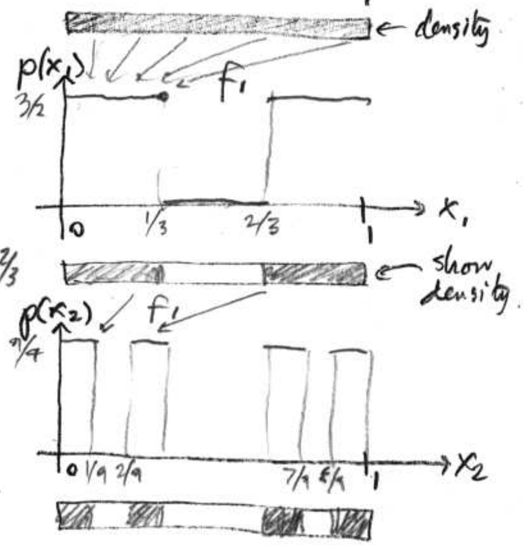
Starting with $p(x_0)$ uniform in $[0,1]$,



Find $p(x_1)$ and sketch

[Hint: what geometrically does f_2 do?] average them





$$p(x_1) = \begin{cases} 3/2 & x_1 < 1/3 \text{ or } x_1 > 2/3 \\ 0 & \text{otherwise} \end{cases}$$

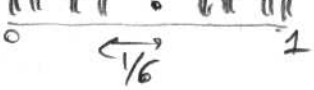
Find $p(x_2)$ and sketch

$$p(x_2) = \begin{cases} 9/4 & x_2 < 1/9 \text{ or } 2/9 < x_2 < 1/3 \text{ or } \\ & 2/3 < x_2 < 7/9 \text{ or } x_2 > 8/9. \\ 0 & \text{otherwise} \end{cases}$$

What is $p(x_n)$? $p(x_n) = \left(\frac{3}{2}\right)^n$ if $x_n \in K_n$, 0 otherwise

What is the limiting attractor set as $n \to \infty$? K_∞, the "missing third" Cantor-Set

Prove an upper bound on the distance of x_n to this set [Hint: dist of $x_0 \le$?]



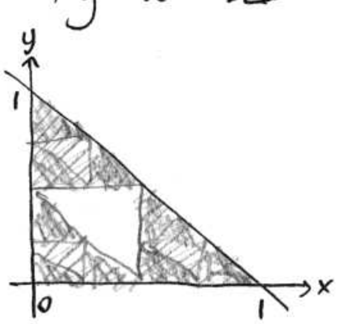
K_∞ worst-case $x_0 = ½$ has $\text{dist}(x_0, K_\infty) = 1/6$

Upon iteration, gets 3 times closer per iteration

$$\Rightarrow \text{dist}(x_n, K_\infty) \le \frac{1}{6 \cdot 3^n}$$

- Now try a 2D example : start with $\vec{x}_0$ uniform in triangle

$\vec{x}_0 = (x_0, y_0)$



Apply $\left\{\begin{array}{l} \vec{f}_1(\vec{x}) = \left(\frac{x}{2}, y/2\right) \\ \vec{f}_2(\vec{x}) = \left(\frac{x+1}{2}, y/2\right) \\ \vec{f}_3(\vec{x}) = \left(x/2, \frac{y+1}{2}\right) \end{array}\right.$ with equal probabilities ⅓.

⇒ geometrically make $\vec{x}$ move half the dist. towards 3 vertices.

← Deduce the attractor set.

Sierpinski gasket for vertices $(0,0)$, $(1,0)$, $(0,1)$.