Modified Lotka-Volterra systems

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“Canonical” Lotka-Volterra system

- Fundamentally: Lotka-Volterra equations are a system of coupled, autonomous, 1\textsuperscript{st}-order ODEs adapted from individual Verhulst (logistic) models:

\begin{align*}
\frac{dx}{dt} &= x(\alpha - \beta y) \\
\frac{dy}{dt} &= y(\delta x - \gamma)
\end{align*}

with \(\alpha, \beta, \gamma, \delta\) all positive constants

**Dynamics of the canonical model:**

Stability: two equilibria at \((0,0)\) and \((\frac{\gamma}{\delta}, \frac{\alpha}{\beta})\)

Jacobians: \(J(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix} \rightarrow \text{SP} \), \(J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{pmatrix} 0 & -\frac{\beta \gamma}{\delta} \\ \frac{a \delta}{\beta} & 0 \end{pmatrix} \rightarrow \lambda = \pm i \sqrt{\alpha \gamma} \rightarrow \text{stable periodic orbits}

Also, as the model is autonomous and 1\textsuperscript{st}-order, Poincaré-Bendixson theorem predicts no chaos for these systems (even in the subsequent variations of these equations)
\[ x'(t) = x(t)(0.3 - 0.2 \cdot y(t)) \]
\[ y'(t) = y(t)(0.4x(t) - 0.5) \]

Initial conditions \((x(0), y(0)) = (3, 1)\)
“Canonical” LV system with 3 species

\[ \frac{dx}{dt} = x(\alpha - \beta y) \]
\[ \frac{dy}{dt} = y(\delta x - \varepsilon z - \gamma) \]
\[ \frac{dz}{dt} = z(\zeta y - \eta) \]

- **System:** \( \frac{dx}{dt} = x(\alpha - \beta y) \quad \frac{dy}{dt} = y(\delta x - \varepsilon z - \gamma) \quad \frac{dz}{dt} = z(\zeta y - \eta) \)
- \( \alpha \): represents the natural growth rate of in the absence of predators
- \( \beta \): represents the effect of predation on \( x \)
- \( \gamma \): represents the natural death rate of \( y \)
- \( \delta \): represents the efficiency rate of \( y \) in the presence of \( x \)
- \( \varepsilon \): represents the effect of predation on species \( y \) by species \( z \)
- \( \zeta \): represents the natural death rate of the predator \( z \) in the absence of prey
- \( \eta \): represents the efficiency of the predator \( z \) in the presence of prey \( y \)
“Canonical” LV system with 3 species

- Equilibria: 2 points: \(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\) and \(\begin{pmatrix} \gamma/\delta \\ \alpha/\beta \\ 0 \end{pmatrix}\).

- Jacobian: 
  \[
  Df = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha - \beta y & -x\beta & 0 \\ y\delta & \delta x - \varepsilon z - \gamma & -y\varepsilon \\ 0 & z\zeta & \zeta y - \eta \end{pmatrix}
  \]

- For parameters \(\alpha=0.3\ \beta=0.2\ \gamma=0.4\ \delta=0.5\ \varepsilon=0.1\ \zeta=0.7\ \eta=0.3\):
  
  For \(Df\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\) are \(\lambda = \pm 0.3, +0.4 \rightarrow\) saddle, and

  For \(Df\begin{pmatrix} \gamma/\delta \\ \alpha/\beta \\ 0 \end{pmatrix}\) are \(\lambda = \pm 0.1095i, 0.75 \rightarrow\) unstable
“Canonical” LV system with 3 species
“Canonical” LV system with 3 species
Modifying the equations (more complex predator-prey behavior)

- We will look at variations to the relative consumption behavior (overconsumption of $x(t)$ by $y(t)$):
  - first when $(\alpha - \beta y)$ is changed to $(\alpha - \beta y^n)$ (explicitly here, case of $n = 2$)
  - **Dynamics:**

    \[
    \begin{align*}
    \frac{dx}{dt} &= x(\alpha - \beta y^n) \\
    \frac{dy}{dt} &= y(\delta x - \gamma)
    \end{align*}
    \]

    Equilibria at $(0,0)$ ($J = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix} \rightarrow \text{SP}), (\frac{\gamma}{\delta}, \pm \sqrt{\frac{\alpha}{\beta}})$ ($J = 
    \begin{pmatrix}
    0 & -2\beta \left(\frac{\gamma}{\delta}\right) \sqrt{\frac{\alpha}{\beta}} \\
    \delta \sqrt{\frac{\alpha}{\beta}} & 0
    \end{pmatrix}
    \rightarrow \lambda \in i\mathbb{R} \rightarrow \text{periodic orbits}$)
And now changing \((\alpha - \beta y)\) to \((\alpha - \beta e^y)\):

**Equilibria at** \((0, 0)\) **and** \(\left(\frac{\gamma}{\delta}, \log \frac{\alpha}{\beta}\right)\)

For \((0, 0)\): \(J = \begin{pmatrix} \alpha - \beta & 0 \\ 0 & -\gamma \end{pmatrix}\) (then SP for \(\alpha > \beta\), sink for \(\alpha < \beta\))

For \(\left(\frac{\gamma}{\delta}, \log \frac{\alpha}{\beta}\right)\): \(J = \begin{pmatrix} 0 & -\frac{\gamma \alpha}{\delta} \\ \delta \log \frac{\alpha}{\beta} & 0 \end{pmatrix}\) (periodic orbits for \(\alpha > \beta\), SP for \(\alpha < \beta\))

\[\alpha > \beta\]

\[x'(t) = x(t)(0.3 - 0.2e^{\gamma(t)})\]

\[y'(t) = y(t)(0.4x(t) - 0.5)\]
\[ \alpha = \beta \]

\[ \alpha < \beta \]