MATH 53 WORKSHEET : Linear torus maps

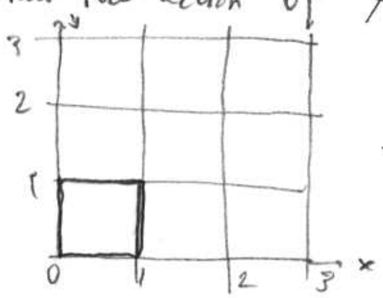
10/17/07
Barnett.

Consider $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \pmod 1$ $\quad a,b,c,d \in \mathbb{Z}$.

[Step 3] Assume A has no eigenvalue equal to 1 (maybe write down the condition this gives for a,b,c,d? ...)

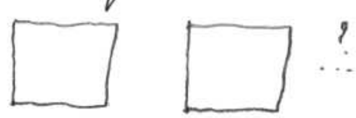
Show that $f(\vec{p}) = \vec{p} \implies \vec{p}$ has <u>rational</u> components $\begin{pmatrix} x \\ y \end{pmatrix}$

[Step 5] Draw the action of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ on the unit square



Show how the pieces rearrange to fill some squares:

how many?



How many squares filled for general A ?

How many solutions are there to $\vec{f}(\vec{x}) = \vec{x}_0$ for a given $\vec{x}_0 \in \mathbb{T}^2$?

Bonus: How many solutions to $f(\vec{x}) = \vec{x}$? [Hint use matrix $A - I$ in above]

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SOLUTIONS

Consider $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \pmod 1$ $\quad a,b,c,d \in \mathbb{Z}$.

[Step 3] Assume A has no eigenvalue equal to 1 (maybe write down the condition this gives for a,b,c,d? ... $\underbrace{\begin{vmatrix} a-1 & b \\ c & d-1 \end{vmatrix}}_{\det(A-1I)} \neq 0$ ie $(a-1)(d-1) - bc \neq 0$)

Show that $f(\vec{p}) = \vec{p} \implies \vec{p}$ has <u>rational</u> components $\begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{cases} ax + by = x + n \\ cx + dy = y + m \end{cases} \quad \text{handles the (mod 1)} \quad n,m \in \mathbb{Z}$$

so

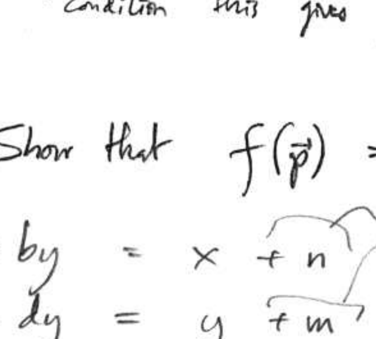
$$(a-1)x + by = n$$
$$cx + (d-1)y = m$$

$$\text{so} \quad c(a-1)x + cby = cn$$
$$\ominus \quad (a-1)cx + (a-1)(d-1)y = (a-1)m$$

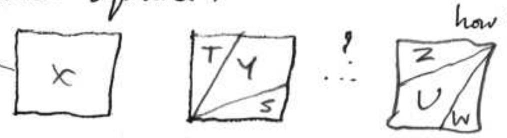
$$y\,\underbrace{[(a-1)(d-1) - bc]}_{\text{since not zero}} = (a-1)m - cn,$$

have $y = \frac{\text{int}}{\text{int}} = \text{rational}$. Same for x.

[Step 5] Draw the action of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ on the unit square



Show how the pieces rearrange to fill some squares:



how many? 3

How many squares filled for general A? $|\det A|$ since $\det A$ gives area expansion factor (lin. alg.).

How many solutions are there to $f(\vec{x}) = \vec{x}_0$ for a given $\vec{x}_0 \in \mathbb{T}^2$?

Well, since there are $|\det A|$ squares filled, there are $|\det A|$ distinct solutions (one from each square)

Bonus: How many solutions to $f(\vec{x}) = \vec{x}$? [Hint use matrix $A - I$ in above].

$f(\vec{x}) - \vec{x} = \vec{0}$ ie $(A - I)\vec{x} = \vec{0}$ our choice for $\vec{x}_0$. So there's $|\det(A-I)|$ fixed points.