

# Attracting Fixed Point Proof

September 13.

Theorem Let  $f$  be a (smooth, i.e.  $f, f'$  are continuous) map and  $p$  a fixed point.

- 1) If  $|f'(p)| < 1$ , then  $p$  is a sink (an attracting fixed point).
- 2) If  $|f'(p)| > 1$ ,  $p$  is a source (a repelling fixed point).

Main Idea: (of 1).

$$\lim_{x \rightarrow p} \frac{|f(x) - f(p)|}{|x - p|} = |f'(p)|$$

we want  $|f^k(x) - p| \rightarrow 0$   
as  $k \rightarrow \infty$  for  $x$  "close to"  $p$ .

$$|f'(p)| < 1 \Rightarrow \underbrace{|f(x) - f(p)|}_{p} < |x - p| \dots$$

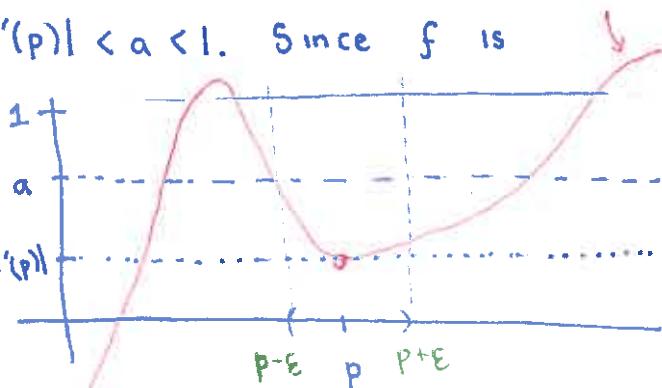
$$\frac{|f(x) - f(p)|}{|x - p|}$$

Pf (of 1).

Fix constant "a" satisfying  $|f'(p)| < a < 1$ . Since  $f$  is continuous, the expression

$$\frac{|f(x) - f(p)|}{|x - p|} \text{ is continuous,}$$

except at  $x=p$ . By the definition of the derivative, we have



$\lim_{x \rightarrow p} \frac{|f(x) - f(p)|}{|x - p|} = |f'(p)|$ , allowing us to sketch the cartoon on the right. Therefore, there is an  $\epsilon > 0$  so that if

$$x \in (p - \epsilon, p + \epsilon), \text{ then } \frac{|f(x) - f(p)|}{|x - p|} < a.$$

Thus, if  $x \in (p - \epsilon, p + \epsilon)$  ( $x \neq p$ ), we have:

$$|f(x) - p| = |f(x) - f(p)| < a|x - p| < a\epsilon < \epsilon$$

$$\begin{matrix} b/c \\ f(p)=p \end{matrix}$$

$$\begin{matrix} b/e \\ \text{previous} \\ \text{line.} \end{matrix}$$

$$\begin{matrix} b/c \\ x \in (p - \epsilon, p + \epsilon). \end{matrix}$$

$$\begin{matrix} b/c \\ a < 1 \end{matrix}$$

Therefore,  $f(x) \in (p - \epsilon, p + \epsilon)$ .

We need to be able to say things about all  $f^k(x) \dots$  so we need induction. Let's summarize what we just found in a fact for induction.

FACT \*: If  $x \in (p-\epsilon, p+\epsilon)$  ( $x \neq p$ ), then  $f(x) \in (p-\epsilon, p+\epsilon)$  ( $f(x) \neq p$ )  
and  
 $|f(x) - p| < a|x - p|$ .

WANT TO SHOW: For all  $k \geq 1$ ,

$$f^k(x) \in (p-\epsilon, p+\epsilon) \text{ and } |f^k(x) - p| < a^k|x - p|.$$

We use proof by induction. The base case  $k=1$  holds by FACT \*.  
For the induction hypothesis, suppose that

$$f^{k-1}(x) \in (p-\epsilon, p+\epsilon) \text{ and } |f^{k-1}(x) - p| < a^{k-1}|x - p|. \quad (*)$$

(We must show the statement holds for  $k$ ).

By FACT \*  $\underbrace{f^{k-1}(x)}_{\text{the pt}} \in (p-\epsilon, p+\epsilon)$  implies that

$$\circ f(f^{k-1}(x)) \in (p-\epsilon, p+\epsilon) \text{ and } |f(f^{k-1}(x)) - p| < a|f^{k-1}(x) - p|.$$

Using the induction hypothesis, we now have,

$$|f^k(x) - p| = |f(f^{k-1}(x)) - p| < a|f^{k-1}(x) - p| < a \cdot a^{k-1}|x - p| = a^k|x - p|. \quad (*)$$

Using the statement for  $(k-1)$ , we showed  $\circ f^k(x) \in (p-\epsilon, p+\epsilon)$   
and  $|f^k(x) - p| < a^k|x - p|$ . By induction, we conclude  
that for all  $k \geq 1$ , if  $x \in (p-\epsilon, p+\epsilon)$ , then

$$|f^k(x) - p| < a^k|x - p|.$$

Since  $a < 1$ , this is saying that  $f^k(x)$  is now within smaller and smaller neighborhoods of  $p$ , (since  $\lim_{k \rightarrow \infty} a^k = 0$ ). Therefore,  
beginning at any  $x \in (p-\epsilon, p+\epsilon)$ ,

$$\lim_{k \rightarrow \infty} |f^k(x) - p| = 0 \quad \text{and so} \quad f^k(x) \rightarrow p \quad \text{as } k \rightarrow \infty.$$

Therefore,  $p$  is an attracting fixed point and attracts this  $\epsilon$ -neighborhood

