For the map $F_2(x) = 2x \bmod 1$, show that x_0 is eventually periodic if and only if x_0 is rational.

Suppose that $x_0 = p/q$ is a rational number between 0 and 1, and so $0 \le p < q$. Notice that
$$F_2(p/q) = 2(p/q) \bmod 1 = \begin{cases} 2p/q & \text{if } p/q \le 1/2 \\ \frac{2p-q}{q} & \text{if } p/q > 1/2 \end{cases}.$$

FIRST DIRECTION

Therefore, $F_2(p/q) = p'/q$, and since $F_2(x) \in [0,1)$, $0 \le p' < q$.

By induction, $F_2^n(p/q) = p^{(n)}/q$ for all $n \ge 0$. Since there are only q distinct values possible for $p^{(n)}$, i.e. $p^{(n)} \in \{0, 1, \dots, q-1\}$ for all $n \ge 0$, we must eventually have a duplicate. That is, there exists some $n \ge 0$ and $k \ge 1$ such that
$$F_2^n(p/q) = F_2^{n+k}(p/q),$$
and so $x_0 = p/q$ is eventually k-periodic. ✓

SECOND DIRECTION

Let x_0 be an eventually periodic orbit. In particular, there exists an $n \ge 0$ and $k \ge 1$ such that $F_2^n(x_0) = F_2^{n+k}(x_0)$. Therefore, the base-2 representation of x_0 must be a word of the form $(\omega_1 \omega_2 \dots \omega_n)\underbrace{(\omega_{n+1} \dots \omega_{n+k})^{\infty}}_{\text{repeating}}$.

Write
$$x_0 = \underbrace{\frac{\omega_1}{2} + \frac{\omega_2}{2^2} + \dots \frac{\omega_n}{2^n}}_{\text{clearly } \mathbb{Q} \text{ b/c common denominator } 2^n} + \underbrace{\left(\frac{\omega_{n+1}}{2^{n+1}} + \frac{\omega_{n+2}}{2^{n+2}} + \dots + \frac{\omega_{n+k}}{2^{n+k}}\right) + \left(\frac{\overset{\omega_{n+1}}{\overset{\shortparallel}{\omega_{n+k+1}}}}{2^{n+k+1}} + \dots + \frac{\overset{\omega_{n+k}}{\overset{\shortparallel}{\omega_{n+2k}}}}{2^{n+2k}}\right) + \dots}_{(*) \text{ must show that this is rational.}}$$

this other piece + periodic piece.

$$(*) \left(\frac{\omega_{n+1}}{2^{n+1}} + \frac{\omega_{n+2}}{2^{n+2}} + \dots + \frac{\omega_{n+k}}{2^{n+k}}\right) + \left(\frac{\omega_{n+1}}{2^{n+k+1}} + \frac{\omega_{n+2}}{2^{n+k+2}} + \dots + \frac{\omega_{n+k}}{2^{n+2k}}\right) + \dots$$
$$= \frac{1}{2^n} \underbrace{\left(\frac{\omega_{n+1}}{2} + \frac{\omega_{n+2}}{2^{n+2}} + \dots \frac{\omega_{n+k}}{2^k}\right)}_{\text{a } \mathbb{Q} \text{ number call it } M}\left(1 + \frac{1}{2^k} + \frac{1}{2^{2k}} + \frac{1}{2^{3k}} + \dots\right) = \frac{1}{2^n} M \sum_{j=0}^{\infty} \left(1/2^k\right)^j = \frac{1}{2^n} M \underbrace{\left(\frac{1}{1 - 1/2^k}\right)}$$

a rational # b/c product of $\mathbb{Q}$. $\Rightarrow$ If eventually periodic, then rational.