

Note: All of these ideas are fair game for the exam. I will not ask you to move between  $(x, y) \leftrightarrow (r, \theta)$ . Nor are you expected to be able to solve #4 independently - although you should know how to show a point is on/not on a stable manifold.

### Limiting Sets and Stable Manifolds for Continuous Dynamical Systems

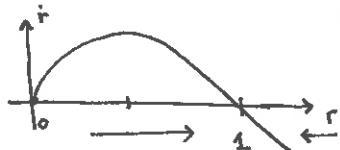
**Definition** A point  $z \in \mathbb{R}^n$  is in the *limit set* of a trajectory  $F(t, v_0)$  if there is a sequence of points increasingly far out along the orbit (i.e. as  $t \rightarrow \infty$ ) which converges to  $z$ .

*Note:* So far, we have seen limit sets that are closed loops and limit sets that are a single point. Poincaré-Bendixson tells us that these are essentially the only two possibilities in  $\mathbb{R}^2$ . In higher dimensions, limit sets can be far more complicated (i.e. strange attractors).

- Find the limit set for

$$\begin{aligned}\dot{r} &= r(1 - r^2) \\ \dot{\theta} &= 1.\end{aligned}$$

determine the long-term behavior.



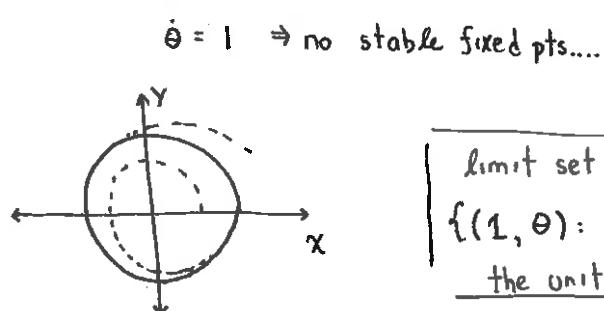
⇒ stable fixed point that attracts all  $r > 1$  at  $r_0 = 1$ .

- Find the limit set for

$$\begin{aligned}\dot{r} &= r(1 - r^2) \\ \dot{\theta} &= 2\sin^2(\theta/2).\end{aligned}$$

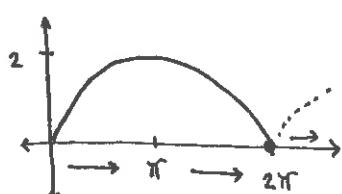
determine the long-term behavior.

stable fixed point that attracts all  $r > 1$  at  $r_0 = 1$ .



limit set  
 $\{(1, \theta) : \theta \in [0, 2\pi]\}$   
 the unit circle.

(as long as initial condition is not  $(0, 0)$ ).



approaching  $\theta = 2\pi$ , we are reaching the attracting fixed point but it is not stable because  $\theta = 2\pi + \epsilon = \epsilon$  must travel around the circle again.

limit set  
 $(r_0, \theta_0) = (1, 0)$ .

(as long as initial condition is not  $(0, 0)$ ).

*Note:* The limit set in the previous question is not a stable equilibrium. Why?

it's not a stable equilibrium b/c it repels for  $\theta > 2\pi$ .

} a semi-stable equilibrium.

Limiting Sets and Stable Manifolds for Continuous Dynamical Systems

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3. Consider the system of equations

$$\dot{x} = x - 2y - x(x^2 + 3y^2)$$

$$\dot{y} = 2x + y - y(x^2 + 3y^2).$$

(a) Classify the fixed point at the origin.

$$Df(0,0) = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}, \quad \begin{aligned} (1-\lambda)(1-\lambda) + 4 &= 0 \\ \lambda^2 - 2\lambda + 5 &= 0 \\ \frac{2 \pm \sqrt{4-20}}{2} &= \frac{2 \pm 4i}{2} = 1 \pm 2i \end{aligned}$$

The fixed point is an unstable spiral.

(b) Rewrite the system of differential equations in polar coordinates.

Use  $r^2 = x^2 + y^2$  and differentiate both sides.

$$\theta = \arctan\left(\frac{y}{x}\right) \rightarrow \begin{cases} 2r\dot{r} = 2x\dot{x} + 2y\dot{y} \\ \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} \end{cases} \quad \begin{cases} \dot{r} = \frac{x\dot{x} + y\dot{y}}{r} \\ \dot{\theta} = 2 \end{cases}$$

$$\dot{r} = \frac{x^2 - 2xy - x^2(x^2 + 3y^2) + 2xy + y^2 - y^2(x^2 + 3y^2)}{r}$$

$$\dot{r} = \frac{r^2 - r^2(r^2 + 2y^2)}{r} \quad y = rsin\theta$$

$$\dot{r} = r - r(r^2 + 2r^2\sin^2\theta) = r(1 - r^2 - 2r^2\sin^2\theta)$$

$$\dot{\theta} = \frac{2x^2 + 2xy - xy(x^2 + 3y^2) - xy + 2y^2 + xy(x^2 + 3y^2)}{r^2} = 2$$

## Limiting Sets and Stable Manifolds for Continuous Dynamical Systems

**Definition** For a fixed point  $x^*$ , the stable manifold  $S(x^*)$  is the set of all points  $x_0$  that tend to the fixed point as  $t \rightarrow \infty$ .

$$S(x^*) = \{x_0 : \lim_{t \rightarrow \infty} |F(t, x_0) - x^*| = 0\}.$$

And similarly, the unstable manifold is the set of points  $x_0$  that tend to the fixed point as  $t \rightarrow -\infty$ ,

$$U(x^*) = \{x_0 : \lim_{t \rightarrow -\infty} |F(t, x_0) - x^*| = 0\}.$$

4. Consider

$$\dot{x}_1 = -x_1$$

$$\dot{x}_2 = -x_2 - x_1^2$$

$$\dot{x}_3 = x_3 + x_1^2.$$

The general solution is given by

$$F(t, v_0) = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{-t} + c_1^2 (e^{-t} + e^{-2t}) \\ c_3 e^t + \frac{c_1^2}{3} (e^t - e^{-2t}) \end{pmatrix},$$

where  $v_0 = (c_1, c_2 + 2c_1^2, c_3)$ . Show that  $v^* = (0, 0, 0)$  is an unstable equilibrium. Then, verify that the stable manifold of  $v^* = (0, 0, 0)$  is

$$S(v^*) = \{(x_1, x_2, x_3) : x_3 = -x_1^2/3\} = \{(q_1, q_2, q_3) : q_3 = -\frac{q_1^2}{3}\}.$$

Just to verify that this is the solution.

$$x_1 = c_1 e^{-t}$$

$$x_2 = c_2 e^{-t} + c_1^2 e^{-t} + c_1^2 e^{-2t}$$

$$x_3 = c_3 e^t + \frac{c_1^2}{3} e^t - \frac{c_1^2}{3} e^{-2t}$$

$$\dot{x}_1 = -c_1 e^{-t} = -x_1$$

$$\begin{aligned} \dot{x}_2 &= -c_2 e^{-t} - c_1^2 e^{-t} - 2c_1^2 e^{-2t} \\ &= -(c_2 e^{-t} + c_1^2 e^{-t} + c_1^2 e^{-2t}) - \underbrace{c_1^2 e^{-2t}}_{(c_1 e^{-t})^2} \\ &= -x_2 - x_1^2. \end{aligned}$$

Notice that the piece that is preventing us from reaching  $(0, 0, 0)$  is when we have  $e^t$  (rather than  $e^{-t}$ ).

If  $(c_3 + \frac{c_1^2}{3}) = 0$ , then  $e^t$  drops out.

which means that if we start at an initial condition  $(a_1, a_2, \frac{-a_1^2}{3})$ ,

$$\left. \begin{aligned} \text{then } a_3 + \frac{a_1^2}{3} &= 0 \text{ and } x_1 = a_1 e^{-t} \\ x_2 &= a_2 e^{-t} + a_1^2 e^{-t} + a_1^2 e^{-2t} \\ x_3 &= a_1^2 / 3 e^{-2t}. \end{aligned} \right\}$$

$$\begin{aligned} x_3 &= c_3 e^t + \frac{c_1^2}{3} e^t + \frac{2}{3} c_1^2 e^{-2t} \\ &= (c_3 e^t + \frac{c_1^2}{3} e^t - \frac{c_1^2}{3} e^{-2t}) + (c_1 e^{-t})^2 \\ &= x_3 + x_1^2. \end{aligned}$$

$$\left. \begin{aligned} \text{as } t \rightarrow \infty \\ (x_1(t), x_2(t), x_3(t)) \rightarrow (0, 0, 0) \end{aligned} \right\}$$

Limiting Sets and Stable Manifolds for Continuous Dynamical Systems

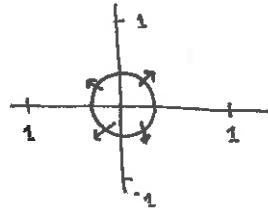
- (c) Find the maximum radius  $r_{\min}$  for the circle on which all the solutions are crossing outward across it.

When are we guaranteed that  $\dot{r} > 0$ ?

$$\dot{r} = r(1 - r^2 - 2r^2 \sin^2 \theta) \stackrel{(*)}{>} r(1 - r^2 - 2r^2) = r(1 - 3r^2) \stackrel{?}{>} 0$$

$$1 > 3r^2 \Rightarrow \frac{1}{3} > r^2 \Rightarrow \boxed{\sqrt{\frac{1}{3}} > r}$$

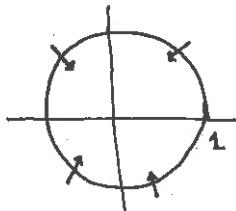
(\*) ask: what is the most we can subtract?  
 $0 \leq \sin^2 \theta \leq 1$



- (d) Find the minimum radius  $r_{\max}$  for the circle on which all the solutions are crossing inward across it.

$$\dot{r} = r(1 - r^2 - 2r^2 \sin^2 \theta) \stackrel{(*)}{<} r(1 - r^2 - 0) \stackrel{?}{<} 0$$

$$1 < r^2 \Rightarrow \boxed{r > 1}$$

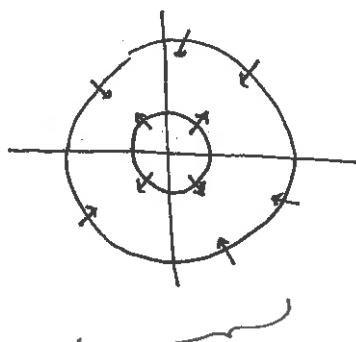


(\*) ask ourselves:  
 what is the least amount we can subtract?  
 $0 \leq \sin^2 \theta \leq 1$

- (e) Prove that there is a periodic orbit somewhere in the annulus  $r_{\min} \leq r \leq r_{\max}$ .

Let  $R$  be the annulus:

$$R = \{(r, \theta) : \sqrt{3}/3 < r < 1, \theta \in [0, 2\pi]\}$$



$R$  is the trapping region.

Notice that  $R$  does not contain any fixed points (the only fixed point for the map is  $(0,0)$ ).

And that all trajectories that enter  $R$  are trapped. By Poincaré-Bendixson, there is a limit cycle in this region.

Limits Sets and Stable Manifolds for Continuous Dynamical Systems

5. Consider

$$\dot{x} = -2x + y$$

$$\dot{y} = 4 - 2xy.$$

Find the fixed points and determine their stability. Indicate the local behavior of the system near each fixed point (saddle, spiral, ...).

$$\begin{aligned} \dot{x} = 0 &\Rightarrow 2x = y \\ \dot{y} = 0 &\Rightarrow 2 = xy. \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad (x_*, y_*) = (1, 2) \quad (x'_*, y'_*) = (-1, -2).$$

$$Df(x, y) = \begin{pmatrix} -2 & 1 \\ -2y & -2x \end{pmatrix}$$

stability @ (1, 2).

$$Df(1, 2) = \begin{pmatrix} -2 & 1 \\ -4 & -2 \end{pmatrix}$$

$$(\lambda+2)(\lambda+2) + 4 = 0$$

$$\Rightarrow \lambda = -2 \pm 2i$$

stable and  
spiral.  
inward.

stability @ (-1, -2).

$$Df(-1, -2) = \begin{pmatrix} -2 & 1 \\ 4 & 2 \end{pmatrix}$$

$$\lambda = \pm 2\sqrt{2}$$

$\underbrace{\phantom{0}}$   
saddle  
point.

Limiting Sets and Stable Manifolds for Continuous Dynamical Systems

6. Consider

$$\begin{aligned}\dot{x} &= -\frac{x}{2} + y \\ \dot{y} &= 1 - y^2.\end{aligned}$$

Determine the fixed points and the stability of each. Indicate the local behavior of the system near each fixed point (saddle, spiral, ...).

fixed points are

$$(x_0, y_0) = (2, 1) \quad \text{and} \quad (x'_0, y'_0) = (-2, -1)$$

$$Df(x, y) = \begin{pmatrix} -1/2 & 1 \\ 0 & -2y \end{pmatrix}$$

stability @ (2, 1)

$$\begin{pmatrix} -1/2 & 1 \\ 0 & -2 \end{pmatrix}$$

$$\lambda = \underbrace{-1/2, -2}_{\text{stable, no spiral.}}$$

stability @ (-2, -1)

$$\begin{pmatrix} -1/2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\lambda = \underbrace{-1/2, 2}_{\text{saddle point, no spiral.}}$$