MATH 54 - TOPOLOGY
SUMMER 2015
FINAL EXAMINATION

ELEMENTS OF SOLUTION

Problem 1

1. Let X be a Hausdorff space and K_1, K_2 disjoint compact subsets of X. Prove that there exist disjoint open sets U_1 and U_2 such that $K_1 \subset U_1$ and $K_2 \subset U_2$.

We know that points can be separated from compact sets in Hausdorff spaces. In other words, for every $x \in K_1$, there exist U_1^x neighborhood of x and U_2^x open containing K_2 such that

$$U_1^x \cap U_2^x = \varnothing.$$

The family $\{U_1^x \ , \ x \in K_1\}$ is an open cover of K_1 compact so we can extract a finite subcover $\{U_1^{x_1} \ \ldots, U_1^{x_n}\}$. Then

$$K_1 \subset \bigcup_{i=1}^{n} U_1^{x_i} \stackrel{\text{def}}{=} U_1,$$

and U_1 is open as the union of open sets. Moreover, $K_2 \subset U_2^{x_i}$ for every $i \in \{1, \ldots, n\}$, so

$$K_2 \subset \bigcap_{i=1}^{n} U_2^{x_i} \stackrel{\text{def}}{=} U_2,$$

open as the finite intersection of open sets. To conclude, observe that U_1 and U_2 are disjoint because $U_1^x \cap U_2^x = \varnothing$ for all x.

2. Let X be a discrete space. Describe the compact subsets of X.

Let K be a compact subset of X. Since the topology is assumed discrete, singletons are open and the family $\{\{x\} \ , \ x \in K\}$ is a covering of K. The fact that a finite subcover can be extracted shows that K must be finite.
The converse holds for non-necessarily discrete topologies so the compact subsets of a discrete space are exactly the finite sets.

Problem 2

A topological space is said *totally disconnected* if its only connected subspaces are singletons.

1. Prove that a discrete space is totally disconnected.

In a discrete space X, singletons are open and closed. Therefore, the connected component of $x \in X$ is $\{x\}$. Another way to see this is to observe that any non-trivial partition of a set is a separation, since all subsets are open and closed in the discrete topology.

2. Does the converse hold?

No, consider the example of $\mathbb{Q}$: any open set of $\mathbb{Q}$ contains a subset of the form $(a, b) \cap \mathbb{Q}$ with $a < b$. Such a set contains infinitely many rationals. In particular, singletons are not open, which means that the topology induced by $\mathbb{R}$ is not discrete.
It is however totally disconnected: for any subset A of $\mathbb{Q}$ containing at least two elements q and r, there exists an irrational z such that $q < z < r$, so that $A \cap (-\infty, z) \sqcup A \cap (z, +\infty)$ is a separation of A.

Problem 3

Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces; let $A_\alpha \subset X_\alpha$ for each $\alpha \in J$.

1. In $\prod_{\alpha \in J} X_\alpha$ equipped with the product topology, prove that

$$\prod_{\alpha \in J} \bar{A}_\alpha = \overline{\prod_{\alpha \in J} A_\alpha}.$$

Let $(x_\alpha)_{\alpha \in J}$ be a closure point of $\prod_\alpha A_\alpha$. And consider, for $\beta \in J$, a neighborhood U_β of x_β. Since the projection maps are continuous, $\pi_\beta^{-1}(U_\beta)$ is open in $\prod_\alpha X_\alpha$, hence a neighborhood of $(x_\alpha)_{\alpha \in J}$ so it contains a point $(y_\alpha)_{\alpha \in J} \in \prod_\alpha A_\alpha$. In particular $y_\beta \in U_\beta \cap A_\beta$ so $x_\beta \in \bar{A}_\beta$.

Conversely, let $(x_\alpha)_{\alpha \in J} \in \prod_\alpha \bar{A}_\alpha$ and $U = \prod_\alpha U_\alpha$ a neighborhood of $(x_\alpha)_{\alpha \in J}$. Then every U_α contains a point $y_\alpha \in U_\alpha \cap A_\alpha$ so $(y_\alpha)_{\alpha \in J} \in U \cap \prod A_\alpha$, which means that $(x_\alpha)_{\alpha \in J} \in \overline{\prod_\alpha A_\alpha}$.

2. Does the result hold if $\prod_{\alpha \in J} X_\alpha$ carries the box topology?

Yes. Observe that the continuity of the projection maps used in the previous question still holds in the box topology. The other part of the proof also carries over without change.

Problem 4

Is $\mathbb{R}$ homeomorphic to $\mathbb{R}^2$?

Assume the existence of a homeomorphism $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^2$ and consider the restricted map $\tilde{\varphi} = \varphi\big|_{\mathbb{R}\setminus\{0\}} : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}^2 \setminus \{f(0)\}$. Then $\tilde{\varphi}$ is bijective by construction and continuous as the restriction of a continuous function. Observe that $\tilde{\varphi}^{-1}$ is continuous for the same reason, which means that $\tilde{\varphi}$ is a homeomorphism between the disconnected space $\mathbb{R} \setminus \{0\}$, and the connected connected space $\mathbb{R}^2 \setminus \{\varphi(0)\}$, which is impossible since connectedness is a topological property.

Problem 5

Let (E, d) be a metric space. An *isometry of* E is a map $f : E \longrightarrow E$ such that

$$d\left(f(x), f(y)\right) = d(x, y)$$

for all $x, y \in E$.

1. Prove that any isometry is continuous and injective.

Let f be an isometry. Then, injectivity follows from the equivalences

$$f(x_1) = f(x_2) \Leftrightarrow d(f(x_1), f(x_2)) = 0 \Leftrightarrow d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2.$$

To prove that f is continuous, let $(x_n)_{n\geq 1}$ be a sequence that converges to $x \in E$. Then

$$\lim_{n\to\infty} d(f(x_n), f(x)) = \lim_{n\to\infty} d(x_n, x) = 0$$

so $\lim\limits_{n\to\infty} f(x_n) = f(x)$, which proves that f is sequentially continuous at any x in E metric.

Assume from now on that E is compact and f an isometry. We want to prove that f is surjective. Assume to the contrary the existence of $a \notin f(E)$.

2. Prove that there exists $\varepsilon > 0$ such that $B(a, \varepsilon) \subset E \setminus f(E)$.

It suffices to prove that $f(E)$ is closed, which follows from the fact that $f(E)$ is compact as the continuous image of E compact. Since E is Hausdorff, any compact in E is closed.

3. Consider the sequence defined by $x_1 = a$ and $x_{n+1} = f(x_n)$. Prove that

$$d(x_n, x_m) \geq \varepsilon$$

for $n \neq m$ and derive a contradiction.

Assume without loss of generality that $1 < n < m$. Then by definition of the sequence,

$$d(x_n, x_m) = d(a, x_{m-n}) = d(a, \underbrace{f(x_{m-n-1})}_{\in f(E)}).$$

Since no point in $f(E)$ is at distance less than ε of a, it follows that $d(x_n, x_m) \geq \varepsilon$.
Now since $(x_n)_{n\geq 1}$ is a sequence in E metric and compact, it admits a subsequence $(u_n)_{n\geq 1}$

with $\lim_{n\to\infty} u_n = \ell \in K$. Then for m and n large enough to guarantee that $d(u_n, k) < \frac{\varepsilon}{2}$ and $d(u_m, k) < \frac{\varepsilon}{2}$, the triangle inequality implies that

$$d(u_n, u_m) \leq d(u_n, k) + d(k, u_m) < \varepsilon,$$

which contradicts the property of $(x_n)_{n\geq 1}$ established above.

4. Prove that an isometry of a compact metric space is a homeomorphism.

we prove that any continous bijection from a compact space to a Hausdorff space is a homeomorphism. Let $f : X \longrightarrow Y$ be such a map. To prove that $g = f^{-1}$ is continuous, it suffices to prove that $g^{-1}(C) = f(C)$ is closed for any C closed in X. Closed subsets of compacts are compact so C is compact, therefore $f(C)$ is compact too since f is continuous. Finally, compact subsets of Hausdorff spaces are closed so $f(C)$ is closed.

Problem 6

Let X be a set, $\mathscr{P}(X)$ the set of subsets of X and $\iota : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ a map satisfying, for all $A, B \subset X$:

$$\begin{array}{lc} (1) & \iota(X) = X \\ (2) & \iota(A) \subset A \\ (3) & \iota \circ \iota(A) = \iota(A) \\ (4) & \iota(A \cap B) = \iota(A) \cap \iota(B). \end{array}$$

1. Check that $A \subset B \;\Rightarrow\; \iota(A) \subset \iota(B)$.

Note that $A \subset B \Leftrightarrow A \cap B = A$, in which case (4) implies $\iota(A) = \iota(A) \cap \iota(B) \subset \iota(B)$.

2. Prove that the family $\mathcal{T} = \{\iota(A)\ ,\ A \in \mathscr{P}(X)\}$ is a topology on X.

Condition (1) says that $X \in \mathcal{T}$. Moreover, (2) implies that $\iota(\varnothing) \subset \varnothing$ so that $\varnothing = \iota(\varnothing)$, which means that $\varnothing \in \mathcal{T}$.
To see that $\mathcal{T}$ is stable under finite intersections, it suffices to prove that the intersection of two elements of $\mathcal{T}$ belongs to $\mathcal{T}$, which is guaranteed by (4), and argue by induction.
Finally, we prove that $\mathcal{T}$ is stable under arbitrary unions. Let $\{A_\alpha\}_{\alpha\in J}$ be a family of subsets of X; we want to prove that $\bigcup_{\alpha\in J} \iota(A_\alpha) = \iota(B)$ for some $B \subset X$. Observe that (2) implies that

$$\bigcup_{\alpha\in J} \iota(A_\alpha) \supset \iota\left(\bigcup_{\alpha\in J} \iota(A_\alpha)\right).$$

Moreover, $\iota(A_\alpha) \subset \bigcup_{\alpha\in J} \iota(A_\alpha)$ for every $\alpha \in J$ so the result proved in (1) gives

$$\iota(A_\alpha) \overset{(3)}{=} \iota\left(\iota(A_\alpha)\right) \subset \iota\left(\bigcup_{\alpha\in J} \iota(A_\alpha)\right).$$

This holds for every $\alpha \in J$ so $\bigcup_{\alpha\in J} \iota(A_\alpha) \subset \iota\left(\bigcup_{\alpha\in J} \iota(A_\alpha)\right)$, hence

$$\bigcup_{\alpha\in J} \iota(A_\alpha) = \iota\left(\bigcup_{\alpha\in J} \iota(A_\alpha)\right).$$

3. Prove that, in this topology, $\mathring{A} = \iota(A)$ for all $A \subset X$.

The definition of $\mathcal{T}$ and (2) imply that $\iota(A)$ is open and a subset of A so $\iota(A) \subset \mathring{A}$. Conversely, $\mathring{(A)}$ is open so it must be of the form $\iota(B)$ for some $B \subset X$. Since $\iota(B) = \mathring{A} \subset A$, it follows from (3) and the result of (1) that

$$\mathring{A} = \iota(B) = \iota(\iota(B)) \subset \iota(A)$$

which concludes the proof.