1. Prove that the L^1 norm is a norm on $\mathbb{R}^n$.

2. (Equivalence of topologies on $\mathbb{R}^n$) Consider the following norms: $\|\cdot\|_2$ (Euclidean norm), $\|\cdot\|_1$, and $\|\cdot\|_\infty$.

   (a) For the case of two metrics d_1 and d_2 on a space X, compare Proposition 4.7 (M13.3) and Lemma M20.2 (which may be used although the proof should be understood).

   (b) (No write-up required.) Using the case of $n = 2$ as an example, show (visually!) that all of the topologies induced by the associated metrics are the same.

   (c) Choose two of the metrics above and consider the topologies they generate on $\mathbb{R}^n$. Choose one of these topologies and prove that it is finer than the other.[1]

   Note that, by Theorem M20.3 and the full result of part (c), each of these norms[2] induce the same topology as the product topology on $\mathbb{R}^n$. From now on, we may switch between these equivalent formulations of the standard topology on $\mathbb{R}^n$ as suits our needs.

3. (Topology on matrices.) Let $M_n(\mathbb{R})$ be the set of all $n \times n$ matrices with entries from $\mathbb{R}$. The goal of this problem is to understand how we can put a topology on $M_n(\mathbb{R})$ analogous to the usual topology in Euclidean space.

   Define the *Hilbert–Schmidt norm* $\|\cdot\|$ on $M_n(\mathbb{R})$ by:

$$\|A\| = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}$$

   for $A = (a_{ij}) \in M_n(\mathbb{R})$ (so a_{ij} is the entry of A in the ith row and jth column). It is a fact that this is, indeed, a norm on $M_n(\mathbb{R})$ and this may be used without proof. The induced metric topology is the *standard topology* on $M_n(\mathbb{R})$.

   (a) Let d be the metric induced by $M_n(\mathbb{R})$. Find 3 examples of matrices in the ball $B_d(I_2, 1)$ where I_2 is the 2×2 identity matrix.

   (b) Prove that $M_n(\mathbb{R})$ is homeomorphic to $\mathbb{R}^{n^2}$.

   Hint: By problem 2, any of the norms in that problem induce the same topology on $\mathbb{R}^{n^2}$. Opt for one of those instead of the product topology.

   (c) Recall the determinant $\det : M_n(\mathbb{R}) \to \mathbb{R}$. Prove that det is a continuous function.

   Hint: Use part (b). To help see how, write down the determinant of a generic element of $M_2(\mathbb{R})$ or $M_3(\mathbb{R})$ and consider the previous homework.

[1] The way that this problem is written should suggest that there are no wrong choices here. In fact, that *is* the case. The conclusion from this is that all three of these topologies are the same.

[2] In fact, a slightly deeper results shows that *any* norm on a finite dimensional vector space induces the same topology.