A Few Repairs

Ok, so I really made a hash of the last example in Monday's lecture. (This came from Munkres Lemma 13.4.)

Let's recap. Let $K = \{ \frac{1}{n} : n \in \mathbf{Z}_+ \}$, and let

$$\begin{aligned} \beta &= \{ (a,b) \subset \mathbf{R} : a < b \} \\ \beta' &= \{ [a,b) \subset \mathbf{R} : a < b \} \\ \beta'' &= \{ (a,b) - K \subset \mathbf{R} : a < b \} \cup \{ (a,b) \subset \mathbf{R} : a < b \}. \end{aligned}$$

Now, thanks to Jacob's altertness—and hence the proper definition of β''—all three of β, β', and β'' cover $\mathbf{R}$—that is, every $x \in \mathbf{R}$ is in some element of β, β', and β''. We just need to verify the intersection property (aka (b)).

Observe that if $x \in (a,b) \cap (c,d)$, then

$$x \in (a',b') \subset (a,b) \cap (c,d)$$

where $a' = \max\{ a,c \}$ and $b' = \min\{ b,d \}$. That is the intersection of two intervals is either empty of another interval. In particular, β is a basis. With the same notation for a' and b',

$$\begin{gathered} ((a,b) - K) \cap ((c,d) - K) = (a',b') - K \quad \text{and} \\ ((a,b) - K) \cap (c,d) = (a',b') - K \end{gathered}$$

provided the intersections are non-empty. It now follows easily that β'' is a basis. On the other hand, if $x \in [a,b) \cap [c,d)$, then $x \in [x,b') \subset [a,b) \cap [c,d)$ where b' is as above. Thus, β' is a basis.

It is immediate from our Proposition on bases, that β is a basis for the usual topology τ on $\mathbf{R}$; that is, $\tau = \tau(\beta)$.

Let $\tau' = \tau(\beta')$ and $\tau'' = \tau(\beta'')$. Munkres calls τ' the lower limit topology and writes $\mathbf{R}_\ell$ for $(\mathbf{R}, \tau')$. He calls τ'' the K-topology and writes $\mathbf{R}_K$ for $(\mathbf{R}, \tau'')$.

We want to prove the following.

Lemma 1. *The three topologies on $\mathbf{R}$—τ, τ', and τ'' are distinct. Moreover $\tau \subsetneq \tau'$ and $\tau \subsetneq \tau''$. But τ' and τ'' are not comparable.*

Proof. Let $U \in \tau$. Suppose $x \in U$. Then there are $a < b$ such that $x \in (a, b) \subset U$. But then $[x, b) \subset U$. This shows that $U \in \tau'$ so that $\tau \subset \tau'$. Since $\beta \subset \beta''$, we clearly have $\tau \subset \tau''$. On the other hand $[0, 1) \in \tau'$ but is not in τ. Hence $\tau \subsetneq \tau'$. Since $0 \in (-1, 1) - K$ and no interval containing 0 lies inside $(-1, 1) - K$, we have $\tau \subsetneq \tau''$. But it is also clear that no basic set of the form $[0, c)$ can be contained in $(-1, 1) - K$ either. Thus $\tau' \subsetneq \tau''$. But $[2, 3) \notin \tau''$, so $\tau'' \subsetneq \tau'$ as well. $\square$