Bases

Definition

If X is a set and $\beta \subset \mathcal{P}(X)$, then we call β a basis in X if

(a) For all $x \in X$ there is a $U \in \beta$ such that $x \in U$.

(b) If $U_1, U_2 \in \beta$ and if $x \in U_1 \cap U_2$, then there is a $U_3 \in \beta$ such that $x \in U_3 \subset U_1 \cap U_2$.

Remark

We can summarize property (a) by saying the union of the elements of β is all of X. We will often just say that β is a cover of X.

Remark

If β covers X and is closed under nonempty intersection—that is $U, V \in \beta$ implies $U \cap V \in \beta$ provided it is nonempty, then (b) holds.

The Generated Topology

Definition

If β is a basis in X, let $\tau(\beta)$ be those subsets $U \subset X$ such that give $x \in U$ there is a $V \in \beta$ such that $x \in V \subset U$.

Theorem

If β is a basis in X, then $\tau(\beta)$ is a topology on X called the topology generated by β. If $U \in \tau(\beta)$, then U is a union of elements from β.

Definition

If (X, τ) is a topological space and β is a basis in X such that $\tau = \tau(\beta)$, then we call β a basis for τ.

Key Proposition

Proposition

Let (X, τ) be a topological space. Suppose $\beta \subset \tau$ is such that give $U \in \tau$ and $x \in U$, then there is a $V \in \beta$ such that $x \in V \subset U$. Then β is a basis for τ.