Metrization

Theorem (Urysohn's Metrization Theorem)

Every second countable regular space is metrizable.

Definition

A sequence (f_n) of functions from X to a metric space (Y, d) converges uniformly to a function $f : X \to Y$ if for all $\epsilon > 0$ there is $N \in \mathbf{Z}_+$ such that $n \geq N$ implies

$$d(f_n(x), f(x)) < \epsilon \quad \text{for all } x \in X.$$

Theorem (Uniform Limit Theorem)

Suppose that (f_n) is a sequence of continuous functions from X into a metric space (Y, d) that converge uniformly to $f : X \to Y$. Then f is continuous.

Remark

Suppose that (f_n) is a sequence of functions from X to $\mathbf{R}$. For each $n \in \mathbf{Z}_+$ let

$$s_n(x) = \sum_{k=1}^{n} f_k(x).$$

Then s_n is continuous if each f_k is. If for all $x \in X$, there is a $f(x)$ such that

$$\lim_n s_n(x) = f(x),$$

then we say that f is the sum of the series $\sum_{k=1}^{\infty} f_k$. We say that $\sum_{k=1}^{\infty} f_k$ converges uniformly to f is (s_n) converges uniformly to f.

Theorem (Weierstrass M-Test)

Suppose that (f_n) is a sequence of continuous functions from X to $\mathbf{R}$ such that

$$|f_k(x)| \leq M_k \quad \text{for all } x \in X.$$

If

$$\sum_{k=1}^{\infty} M_k < \infty,$$

then the series $\sum_{k=1}^{\infty} f_k$ converges uniformly to a necessarily continuous function $f : X \to \mathbf{R}$.