Perfection

Definition

A topological space is perfectly normal if it is normal and every closed subset is a G_δ-set.

Theorem

Suppose that A and B are disjoint closed G_δ-subsets of a normal topological space X. Then there is a continuous function $f : X \to [0,1]$ such that $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$.

Cauchy Sequences

Definition

A sequence (x_n) in a metric space (X, d) is called Cauchy if for all $\epsilon > 0$ there is a $N \in \mathbf{Z}_+$ such that $m, n \geq N$ implies that $d(x_n, y_m) < \epsilon$.

Example

1. Convergent sequences are Cauchy.
2. If a Cauchy sequence has a convergent subsequence, then it converges.
3. If X is metrizable, then whether or not a sequence (x_n) in X is Cauchy can depend on the choice of metric!

Completeness

Definition

We say that a metric space (X, d) is complete if every Cauchy sequence in X is convergent.

Example

1. Every compact metric space is complete.
2. For all $n \in \mathbf{Z}_+$, both $(\mathbf{R}^n, d)$ and $(\mathbf{R}^n, \rho)$ are complete where d is the (usual) Euclidean metric and ρ is the square metric.

Definition

A topological space X is called completely metrizable if X admits a metric d that generates the given topology on X and such that (X, d) is complete. Note that Munkres uses the term topologically complete to mean the same thing.

To Baire All

Definition

A subset D of a topological space X is dense if $\overline{D} = X$.

Lemma (Exercise)

Suppose that $A \subset X$. Then $\operatorname{int}(A) = \emptyset$ if and only if $D = X \setminus A$ is dense.

Proposition

Let X be a topological space. Then the following are equivalent.

- (a) *If U_n is open and dense in X for all $n \in \mathbf{Z}_+$, then $\bigcap_n U_n$ is dense.*
- (b) *If A_n is closed with $\operatorname{int}(A_n) = \emptyset$, then $\operatorname{int}\left(\bigcup_n A_n\right) = \emptyset$.*