Baire Spaces

Definition

A topological space is called a Baire space if the countable intersection of open dense subsets is dense. Alternatively, a space is a Baire space if the countable union of closed sets with empty interior has empty interior.

Example

1. The space $\mathbf{Q} \subset \mathbf{R}$ is not a Baire space.
2. $\mathbf{Z}_+$ is a Baire space.
3. The irrationals, $\mathcal{N} = \mathbf{R} \setminus \mathbf{Q}$ is a Baire space. (This is not obvious and requires the next result to establish.)

The Baire Category Theorem

Theorem (Baire Category Theorem)

If X is a locally compact space or if X is completely metrizable, then X is a Baire space.

Lemma

An open subspace of a Baire space is a Baire space.

Question of the Day

Remark (Question)

Suppose that f is the pointwise limit of continuous functions $f_n : [0,1] \to \mathbf{R}$. We know that f need not be continuous on all of $[0,1]$, but just how pathological can f be? The answer is "not that bad".

Theorem

Suppose that X is a Baire space and that Y is metrizable. For each $n \in \mathbf{Z}_+$, let $f_n : X \to Y$ be a continuous function such that $f_n(x) \to f(x)$ for all $x \in X$. Then the set A of points x in X such that f is continuous at x contains a dense G_δ-set.

Finishing the Proof

Proof of the Theorem.

We let $A_N(\epsilon) = \{\, x \in X : |f_n(x) - f_m(x)| \leq \epsilon \text{ for all } n, m \geq N \,\}$. Then $A_N(\epsilon)$ is closed and $X = \bigcup_{N \in \mathbf{Z}_+} A_N(\epsilon)$. We let

$$U(\epsilon) = \bigcup_{N=1}^{\infty} \operatorname{int}\big(A_N(\epsilon)\big) \quad \text{and} \quad C = \bigcap_{k=1}^{\infty} U\Big(\frac{1}{k}\Big).$$

It will suffice to establish that

1. $U(\epsilon)$ is open and dense in X, and
2. $a \in C$ implies that f is continuous at a.

□