Continuous Functions on $\mathbf{R}$

Theorem

Suppose that $f : \mathbf{R} \to \mathbf{R}$ is any function. Then the set C of points $a \in \mathbf{R}$ where f is continuous is a G_δ-subset of $\mathbf{R}$.

Theorem

Let X be a T_1 Baire space with no isolated points. If $D \subset X$ is countable and dense, then D is not a G_δ-subset of X.

Example

The ruler function $f : \mathbf{R} \to \mathbf{R}$ is continuous at $a \in \mathbf{R}$ if and only if $a \in \mathcal{N} := \mathbf{R} \setminus \mathbf{Q}$.

Corollary

There is no function $f : \mathbf{R} \to \mathbf{R}$ such that f is continuous at $a \in \mathbf{R}$ if and only if $a \in \mathbf{Q}$.

Analysis Again

Theorem

The set $C([0,1])$ *of continuous functions* $f : [0,1] \to \mathbf{R}$ *is a complete metric space with respect to the metric*

$$\rho(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Remark

Convergence in $\big(C([0,1]), \rho\big)$ is the same as uniform convergence on $[0,1]$.

One-Sided Derivatives

Definition

If $f \in C([0,1])$, then we define the one-sided derivatives

$$D^+ f(x) = \lim_{h \searrow 0} \frac{f(x+h) - f(x)}{h} \quad \text{and}$$
$$D^- f(x) = \lim_{h \nearrow 0} \frac{f(x+h) - f(x)}{h}$$

provided the limits exist. We say that $f'(x_0)$ exists whenever the appropriate one-sided derivatives exist and are equal when both are defined.

Differentiable at at Least One Point

Lemma

Suppose that $f \in C\big([0,1]\big)$ *and that* $f'(x_0)$ *exists for some* $x_0 \in [0,1]$. *Then there is a* $n \in \mathbf{Z}_+$ *such that*

$$|f(x) - f(x_0)| \leq n|x - x_0| \quad \textit{for all } x \in [0,1].$$

Definition

For each $n \in \mathbf{Z}_+$ let $\mathcal{F}_n$ be the set of $f \in C([0,1])$ such that there is a $x_f \in [0,1]$ such that

$$|f(x) - f(x_f)| \leq n|x - x_f| \quad \text{for all } x \in [0,1].$$

Remark

Note that if $f \in C([0,1])$ and $f'(x_0)$ exists for at least one point in $[0,1]$, then f is in some $\mathcal{F}_n$.

Getting to Work

Lemma

Suppose that $(f_n) \subset C([0,1])$ *converges uniformly to* f *on* $[0,1]$. *Then if* $x_n \to x$ *in* $[0,1]$, *we have* $f_n(x_n) \to f(x)$.

Lemma

For each $n \in \mathbf{Z}_+$, $\mathcal{F}_n$ *is closed in* $C([0,1])$.