Example

The countable product $\mathbf{R}^\omega$ with the uniform topology is an example of a first countable space which is not second countable.

Theorem

1. *A subspace of a first (second) countable space is first (second) countable.*
2. *A countable product of first (second) countable spaces is first (second) countable.*

Second Countable

Definition

A subspace A of a topological space X is dense in X if $\overline{A} = X$.

Theorem

Suppose X is second countable.

1. *Every open cover of X has a countable subcover.*
2. *X has a countable dense subset.*

Definition

Suppose that points are closed in X.

1. We say that X is regular if whenever A is closed in X and $x \notin A$, then there are disjoint open sets U and V such that $x \in U$ and $F \subset V$.
2. We say that X is normal if given disjoint closed sets A and B in X, then there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

Remark

Note that normal spaces are regular, and that regular spaces are Hausdorff. (This is part of the reason we insist that points be closed in regular and normal spaces.)

Basic Lemma

Theorem (Basic Lemma)

Suppose that points are closed in X.

1. *X is regular if and only if given a neighborhood U of x, there is a neighborhood V of x such that*

$$x \in V \subset \overline{V} \subset U.$$

2. *X is normal if and only if given a neighborhood U of a closed set A, there is a neighborhood V of A such that*

$$A \subset V \subset \overline{V} \subset U.$$

Low Hanging Fruit

Corollary

Every locally compact Hausdorff space is regular.

Theorem

A subspace of a regular space is regular as is the arbitrary product of regular spaces.