Definition

If d is a metric on X, then the metric topology on X induced by d is the topology generated by the basis

$$\beta = \{ B_d(X, \epsilon) : x \in X \text{ and } \epsilon > 0 \}.$$

The set X endowed with this topology is called the metric space (X, d).

Remark

If (X, d) is a metric space, then U is open in X if and only if given $x \in U$ there is a $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U$.

Metrizable Spaces

Definition

A topological space (X, τ) is said to be metrizable if there is a metric d on X for which the induced topology is τ.

Example

If X is any set, then the discrete topology on X is metrizable and is induced by the discrete metric

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y, \text{ and} \\ 1 & \text{if } x \neq y. \end{cases}$$

Equivalent Metrics

Proposition

Suppose that d and d' are metrics on X inducing topologies τ_d and $\tau_{d'}$, respectively. Then $\tau_d \subset \tau_{d'}$ (Munkres says "$\tau_{d'}$ is finer than τ_d") if for all $x \in X$ and all $\epsilon > 0$, there is a $\delta > 0$ such that

$$B_{d'}(x, \delta) \subset B_d(x, \epsilon).$$

Theorem

Let (X, d) be a metric space. Then $\overline{d}(x, y) = \min\{\, d(x, y), 1 \,\}$ is a metric on X that induces the same topology as does d.

Euclidean Space

Theorem

The Euclidean metric $d(x,y) = \|x-y\|_2$, the square metric $\rho(x,y) = \|x-y\|_\infty$, and the diamond metric $\sigma(x,y) = \|x-y\|_1$ all induce the product topology on $\mathbf{R}^n$.

Example

Let $d(x,y) = \rho(x,y) = \sigma(x,y) = |x-y|$ be the usual metric on $\mathbf{R}$. Then

$$\bar{d}(x,y) = \begin{cases} |x-y| & \text{if } |x-y| < 1, \text{ and} \\ 1 & \text{of } |x-y| \geq 1. \end{cases}$$