Uniform metric

Definition

Let $\mathbf{R}^J = \prod_{j \in J} \mathbf{R}$. Then the uniform metric on $\mathbf{R}^J$ is given by

$$\overline{\rho}(x, y) = \sup\{\, \overline{d}(x_j, y_j) : j \in J \,\}$$

where

$$\overline{d}(a, b) = \begin{cases} |a - b| & \text{if } |a - b| < 1, \text{ and} \\ 1 & \text{if } |a - b| \geq 1. \end{cases}$$

The corresponding metric topology on $\mathbf{R}^J$ is called the uniform topology.

Proposition

Let τ_p, τ_u, and τ_b be, respectively, the product topology, the uniform topology, and the box topology on $\mathbf{R}^J$. Then

$$\tau_p \subset \tau_u \subset \tau_b.$$

Metrizable

Theorem

The product topology on $\mathbf{R}^\omega = \prod_{n\in\mathbf{Z}_+}$ *is metrizable. In particular,*

$$D(x,y) = \sup\left\{ \frac{\overline{d}(x_n,y_n)}{n} : n \in \mathbf{Z}_+ \right\}$$

is a metric on $\mathbf{R}^\omega$ *inducing the product topology.*

Theorem

Suppose that (X, d) *and* (Y, ρ) *are metric spaces and* $f : X \to Y$ *is a function. Then* f *is continuous if and only if for all* $x \in X$ *and all* $\epsilon > 0$ *there is a* $\delta > 0$ *such that*

$$d(x, y) > \delta \quad \textit{implies} \quad \rho\big(f(x), f(y)\big) < \epsilon.$$

The Sequence Lemma

Theorem (The Sequence Lemma)

Suppose that A is a subset of a topological space X.

a. *If there is a sequence $(a_n) \subset A$ such that $a_n \to x$ in X, then $x \in \overline{A}$.*
b. *If X is metrizable and $x \in \overline{A}$, then there is a sequence $(a_n) \subset A$ such that $a_n \to x$.*