Connected Spaces

Definition

Let X be a topological space. Then a separation of X is a pair of nonempty disjoint open sets C and D such that $X = C \cup D$. We say that X is connected if here are no separations of X.

Remark

Note that X is connected if the only subsets of X that are both open and closed (a.k.a., clopen sets) are the empty set and the whole space.

Basic Observations

Lemma

Suppose that Y is a connected subspace of X. If $\{ C, D \}$ is a separation of X, then either $Y \subset C$ or $Y \subset D$.

Theorem

Suppose that A_j is a connected subspace of X for all $j \in J$. If $\bigcap_{j \in J} A_j \neq \emptyset$, then $\bigcup_{j \in J} A_j$ is connected.

The Closure of a Connected Set is Connected

Lemma

Suppose that Y is a subspace of X and that A and B are disjoint subset of Y such that $Y = A \cup B$. Then $\{ A, B \}$ is a separation of Y if and only if A contains no limit points of A and B contains no limit points of A.

Remark

Since $A \cap B = \emptyset$, A contains no limit points of B if and only if $A \cap \overline{B} = \emptyset$. Similarly, B contains no limit points of A if and only if $B \cap \overline{A} = \emptyset$.

Theorem

Suppose that A is a connected subset of a topological space X. Then if $A \subset B \subset \overline{A}$, B is connected. In particular, $\overline{A}$ is connected.