Continuous Images of Connected Spaces

Theorem

If A is a connected subset of topological space X and if $f : X \to Y$ is continuous, then $f(A)$ is connected in Y.

Lemma

If X and Y are connected, then so is $X \times Y$. More generally, if $X_1, \ldots, X_n$ are connected, then so is their Cartesian product $\prod_{k=1}^{n} X_k = X_1 \times \cdots \times X_n$.

Corollary (Homework based on Ex 7 in §23)

Suppose that X_α is connected for every $\alpha \in A$, then $\prod_{\alpha \in A} X_\alpha$ is connected.

What About $\mathbf{R}$

Definition

An ordered set $(X, <)$ has the least upper bound property if every nonempty subset A of X which is bounded above has a least upper bound called the supremum of A or $\sup A$.

Definition

An ordered set $(L, <)$ is called a linear continuium if

a. L has the least upper bound property, and
b. given $x < y$ in L there is a $z \in L$ such that $x < z < y$.

Continuiums are Connected

Theorem

Let L be a linear continuium with the order topology. Then L as well as any interval or ray in L is connected.

Corollary

The real line $\mathbf{R}$ *is connected as is any interval or ray in* $\mathbf{R}$.

The Intermediate Value Theorem

Theorem (The Intermediate Value Theorem)

Suppose that X a connected topological space and that Y is an ordered space with the order topology. Suppose that $f : X \to Y$ is continuous and that $a, b \in X$ and $y \in Y$ are such that

$$f(a) < y < f(b).$$

Then there is a $c \in X$ such that

$$f(c) = y.$$