Compact Subsets

Theorem

Suppose that K is a subset of a topological space X.

1. *If X is compact and K is closed, then K is compact.*
2. *If X is Hausdorff and K is compact, then K is closed.*

Theorem

Suppose that X is Hausdorff, that K is a compact subspace, and that $x \notin K$. Then there are disjoint open sets U and V such that $K \subset U$ and $x \in V$.

Theorem

Suppose that $f : X \to Y$ is continuous and that K is a compact subspace of X. Then $f(K)$ is compact

The Tube Lemma

Theorem (The Tube Lemma)

Suppose that X and Y are topological spaces with Y compact. Suppose that N is a neighborhood of $\{x_0\} \times Y$ in $X \times Y$. Then there is a neighborhood W of x_0 such that $W \times Y \subset N$.

Theorem

If X and Y are compact, then so is their product $X \times Y$.

Theorem

The finite product of compact spaces is compact.

The Finite Intersection Property

Definition

A collection $\mathcal{C} = \{ A_j \}_{j \in J}$ has the finite intersection property (FIP) if given any finite subset $F \subset J$, we have

$$\bigcap_{j \in F} A_j \neq \emptyset.$$

Theorem

A topological space X is compact if and only if any collection $\mathcal{C} = \{ A_j \}_{j \in J}$ of closed sets with the FIP also satisfies

$$\bigcap_{j \in J} A_j \neq \emptyset.$$