Compact Sets

Theorem

Suppose that X is an ordered set with the order topology. If X has the least upper bound property, then every closed interval in X is compact.

Corollary

Every closed interval in $\mathbf{R}$ is compact.

Corollary

The ordered square I_0^2 is compact.

Theorem

A subspace A of $\mathbf{R}^n$ is compact if and only if A is closed and bounded with respect to the Euclidean metric $d(x, y) = \|x - y\|_2$.

The Extreme Value Theorem

Theorem (The Extreme Value Theorem)

Suppose that $f : X \to Y$ is continuous with X compact and Y an ordered space with the order topology. Then there are $c, d \in X$ such that

$$f(c) \leq f(x) \leq f(d) \quad \text{for all } x \in X.$$

The Distance Function

Definition

Suppose that (X, d) is a metric space and that $A \subset X$ is not empty. Then for each $x \in X$ we define

$$d(x, A) = \inf\{\, d(x, a) : a \in A \,\}.$$

Lemma

If A is a nonempty subset of a metric space (X, d), then

$$x \mapsto d(x, A)$$

is continuous from X to $\mathbf{R}$.

Lebesgue Numbers

Theorem (The Lebesgue-Number Lemma)

Suppose that (X, d) *is a compact metric space and that* $\mathcal{A} = \{ A_j \}_{j \in J}$ *is an open cover of* X*. Then there is a* $\delta > 0$ *such that given any subset* $A \subset X$ *such that* $\operatorname{diam}(A) < \delta$ *there is a* $j \in J$ *such that* $A \subset A_j$*. We call* δ *a* ***Lebesgue number*** *for the cover* $\mathcal{A}$*.*