Metrizable Spaces

Theorem

Suppose that X is metrizable. Then the following are equivalent.

a. *X is compact.*
b. *X is limit point compact.*
c. *X is sequentially compact.*

Well-Ordered Sets

Definition

An ordered set $(A, <)$ is called well-ordered if every non-empty subset $S \subset A$ has a smallest element.

Example

a. $\mathbf{Z}_+$ in its usual ordering.
b. $\mathbf{Z}_+ \times \mathbf{Z}_+$ in the dictionary order.
c. $\mathbf{Z}_+ \times \mathbf{Z}_+ \times \mathbf{Z}_+$ in the dictionary order.
d. Neither $\mathbf{Z}$ nor $\mathbf{R}$ is well-ordered in their usual orders.

Definition

Of A is a well-ordered set and $\alpha \in A$, then

$$S_\alpha = \{\, x \in A : x < \alpha \,\}$$

is called the section of A by α.

Theorem (Lemma 10.2)

There is a well-ordered set A having a largest element Ω such that the section S_Ω is uncountable, but every other section S_α with $\alpha \neq \Omega$ is countable. We'll always equip S_Ω with the order topology.

Remark

Note that S_Ω has no largest element—if α was a largest element, then $S_\Omega = S_\alpha \cup \{ \alpha \}$ would be countable. Therefore

$$S_\Omega = \bigcup_{a \in S_\Omega} S_a = \bigcup_{a \in S_\Omega} (-\infty, a)$$

is an open cover of S_Ω in the order topology without a finite subcover. Therefore S_Ω is not compact.

Low Hanging Fruit

Lemma

Every countable subset of S_Ω *is bounded.*

Lemma

S_Ω *has the least upper bound property.*

Lemma

We will write $\overline{S}_\Omega$ *for the set* $A = S_\Omega \cup \{\Omega\}$*. Then* $\overline{S}_\Omega$ *has the least upper bound property and hence is compact.*

Non-Metric Compactness is Complicated

Lemma

If X is sequentially compact, then X is limit point compact.

Theorem

S_Ω is sequentially compact (and hence limit point compact).

Corollary

Neither S_Ω nor $\overline{S}_\Omega$ is metrizable.

Corollary

In a general setting, sequential compactness does not imply compactness.

Remark

It is true that $X = [0,1]^\omega = \prod_{n \in \mathbf{Z}_+} [0,1]$ is compact (see Tychonoff's Theorem in §37), but not sequentially compact. So in general, compactness does not imply sequential compactness.

Remark

Note that $(n,m) \mapsto (1,n,m)$ is an order isomorphism of $\mathbf{Z}_+ \times \mathbf{Z}_+$ onto the section $S_{(2,1,1)}$ in $Z_+ \times Z_+ \times Z_+$. This is a general phenomena. If X and Y are well ordered sets, then exactly one of the following holds: X and Y are order isomorphic, or X is order isomorphic to a section of Y, or Y is order isomorphic to a segment of X.

Corollary

Every countable well-ordered set X is order isomorphic to a section of S_Ω.