Limit Points

Definition

Let A be a subset of a topological space X. We say that $x \in X$ is a limit point of A if every neighborhood of x meets $A \setminus \{ x \}$. The set of limit points of A is denoted by A'.

Theorem

Of A is a subset of a topological space X then

$$\overline{A} = A \cup A'.$$

Corollary

If A is closed, then $A' \subset A$.

Hausdorff Spaces

Definition

A topological space X is called Hausdorff if distinct points have disjoint neighborhoods.

Theorem

If X is Hausdorff, then every finite subset of X is closed.

Sequences

Definition

Suppose that (x_n) is a sequence in a topological space X. Then we say that (x_n) converges to $x \in X$ if given any neighborhood U of x there is a $N \in \mathbf{Z}_+$ such that $n \geq N$ implies that $x_n \in U$. Then we write $x_n \to x$ or $\lim_n x_n = x$.

Remark

Alternatively, we say that (x_n) converges to x if (x_n) is eventually in every neighborhood of x.

Theorem

If X is Hausdorff and (x_n) is a sequence in X converging to both x and y, then $x = y$.

Continuous functions

Definition

Suppose that X and Y are topological spaces. Then we say that a function $f : X \to Y$ is continuous if $f^{-1}(V)$ is open in X whenever V is open in Y.

Proposition

Suppose that X and Y are topological spaces and that β is a basis for the topology on Y. Then $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open for every $V \in \beta$.

Continuity at a Point

Definition

Suppose that X and Y are topological spaces and that $f : X \to Y$ is a function. We say that f is continuous at $x_0 \in X$ if given a neighborhood V of $f(x_0)$ there is a neighborhood U of x_0 such that $U \subset f^{-1}(V)$.

Theorem

If X and Y are topological spaces and $f : X \to Y$ is a function, then the following are equivalent.

1. *f is continuous.*
2. *For all $A \subset X$, we have $f(\overline{A}) \subset \overline{f(A)}$.*
3. *$f^{-1}(B)$ is closed in X whenever B is closed in Y.*
4. *f is continuous at every $x \in X$.*