Midterm Exam

1. Of Midterm exam will be tomorrow (Thursday, August 1^{st}) during our x-hour (1:20 to 2:10) with the take home due at the beginning of class on Friday.
2. The exam will cover through Firday's lecture (that is up to an including Section 28).

Local Compactness

Definition

A topological space X is called locally compact if every point in x has a neighborhood contained in a compact subspace of X.

Example

1. Any compact space is locally compact.
2. $\mathbf{R}$ and $\mathbf{R}^n$ are locally compact for all $n \in \mathbf{Z}_+$.
3. $\mathbf{R}^\omega$ is NOT locally compact in the product topology.

The Hausdorff Case

Theorem

Suppose that X is Hausdorff. Then X is locally compact if and only if every point in X has a neighborhood with compact closure.

Theorem (To be Fixed)

Suppose that X is Hausdorff and locally compact. If U is a neighborhood of $x \in X$, then there is a neighborhood V of x with compact closure such that

$$x \in V \subset \overline{V} \subset U.$$

Subspaces

Theorem

Suppose that X is locally compact and Hausdorff. Then a subspace $Y \subset X$ is locally compact if and only if Y is locally closed. In particular, both open and closed subsets of locally compact Hausdorff spaces are themselves locally compact.

First Countability

Definition

If X is a topological space, then X has a countable basis at $x \in X$ if there is a countable collection β of neighborhoods of x such that given any neighborhood U of x there is a $V \in \beta$ such that

$$x \in V \subset U.$$

We say that X is first countable if every point in X has a countable basis.

Example

A metrizable space is always first countable.

Sequences and First Countability

Theorem

Suppose that X is first countable.

1. *If $A \subset X$, then $x \in \overline{A}$ if and only if there is a sequence $(x_n) \subset A$ such that $x_n \to x$.*
2. *A function $f : X \to Y$ is continuous if and only if given any sequence (x_n) in X converging to $x \in X$, then $f(x_n) \to f(x)$.*

Second Countability

Theorem

We say that a topological space X is second countable if there is a countable basis for the topology on X.

Example

$\mathbf{R}$, $\mathbf{R}^n$, and $\mathbf{R}^\omega$ are all second countable.

Lemma

Every second countable space is first countable.