J-tuples

Definition

Let X and J be sets. Then a J-tuple of elements in X is a function $x : J \to X$. When thinking of x as a J-tuple, we often write x_j in place of $x(j)$ and even replace x with $(x_j)_{j \in J}$. We refer to x_j as the "j^{th} coordinate of x.

Example

The terminology and notation come from the case $J = \{1, 2, \ldots, n\}$ and $X = \mathbf{R}$. Then a J-tuple is just he usual n-tuple of real numbers.

Example

If $J = \mathbf{Z}_+$, then a J-tuple is just a sequence in X.

Cartesian Products

Definition

Let $\{ X_i \}_{j\in J}$ be an indexed family of sets. Let $X = \bigcup_{j\in J} X_j$. Then the Cartesian Product

$$\prod_{j\in J} X_j$$

is the set of J-tuples $x = (x_j)$ such that $x_j \in X_j$ for all $j \in J$.

Example

If $J = \{ 1, 2, \ldots, n \}$, then we write

$$\prod_{j\in J} X_j = \prod_{j=1}^{n} X_j = X_1 \times X_2 \times \cdots \times X_n.$$

In particular, $\mathbf{R}^n = \prod_{j=1}^{n} \mathbf{R}$ and $A \times B = \prod_{j\in\{ 1,2 \}} X_j$ where $X_1 = A$ and $X_2 = B$.

The Projection Maps

Definition

Suppose that $X = \prod_{j \in J} X_j$ and that $k \in J$. Then the map

$$\pi_k : \prod_{j \in J} \to X_k$$

given by $\pi_k(x) = x_k$ is the projection onto the k^{th} factor.

Remark

If U_k is open in X_k, then $\pi_k^{-1}(U_k) = \prod U_j$ where $U_j = U_k$ if $j = k$, and such that $U_j = X_j$ if $j \neq k$. In particular,

$$\rho = \{\, \pi_j^{-1}(U_j) : U_j \text{ is open in } X_j \,\}$$

is a subbasis for $X = \prod_{j \in J} X_j$.

The Product Topology

Definition

If X_j is a topological space for each $j \in J$, then the product topology on $\prod_{j \in J} X_j$ is the topology generated by the subbasis ρ above.

Remark

The product topology is the smallest topology on $\prod X_j$ such that all the projection maps are continuous.

Remark

A basis for the product topology is given by sets of the form

$$U = \prod_{j \in J} U_j$$

where there is a finite set $F \subset J$ such that U_j is open in X_j for all $j \in F$ and $U_j = X_j$ if $j \notin F$.

The Box Topology

Definition

The box topology on $\prod X_j$ is the topology generated by the basis consisting of sets of the form

$$U = \prod U_j$$

where U_j is open in X_j for all $j \in J$.

Useful Theorems

Theorem

Let $\prod X_j$ have the product topology and suppose that $f : Z \to \prod X_j$ is a function of the form $f(z) = \big(f_j(z)\big)$ for functions $f_j : Z \to X_j$. Then f is continuous if and only if each f_j is continuous.

Theorem

Suppose that $A_j \subset X_j$ for each j. Then in either the box or product topology,

$$\prod_{j \in J} \overline{A_j} = \overline{\prod_{j \in J} A_j}.$$

Metric Spaces

Definition

A metric on a set X is a function

$$d : X \times X \to [0, \infty)$$

such that for all $x, y, z \in X$ we have

a. $d(x, y) = 0$ if and only if $x = y$,
b. $d(x, y) = d(y, x)$, and
c. $d(x, z) \leq d(x, y) + d(y, z)$.

Definition

If d is a metric on X, $x \in X$, and $\epsilon > 0$, then

$$B_d(x, \epsilon) = \{ y \in X : d(y, x) < \epsilon \}$$

is called the ϵ-ball centered at x.