Order Topology

Definition

Let $(X, <)$ be an ordered set. Then the order topology on X is the topology generated by the basis β consisting of unions of sets of the form

1. Open intervals of the form (a, b) with $a < b$ in X.
2. If X has a smallest element a_0, then we also include half-open intervals $[a_0, b)$ with $a_0 < b$ in X.
3. If X has a largest element b_0, then we also include half-open intervals of the form $(a, b_0]$ with $a < b_0$.

Example

1. The order topology on $\mathbf{R}$ is the usual topology.
2. The order topology on $\mathbf{Z}_+$ is the discrete topology.

The Product Topology

Definition

Then the product topology on the cartesian product $X \times Y$ is the topology generated by the basis of open rectangles

$$\beta = \{ U \times V \subset X \times Y : U \in \tau \text{ and } V \in \sigma \}.$$

Theorem

Let (X, τ) and (Y, σ) be topological spaces. Suppose that β is a basis for τ and γ is a basis for σ. Then

$$\gamma = \{ U \times V \subset X \times Y : U \in \beta \textit{ and } V \in \gamma \}$$

is a basis for the product topology on $X \times Y$.

Example

The product topology on $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ is the usual topology on $\mathbf{R}^2$.

The Subspace Topology

Theorem

Suppose that (X, τ) is a topological space and that $Y \subset X$. Then

$$\tau_Y = \{ U \cap Y : U \in \tau \}$$

is a topology on Y called the subspace topology on Y. We say that (Y, τ_Y) is a subspace of (X, τ)

Theorem

Suppose that Y is a subspace of X. If β is a basis for the topology on X, then

$$\beta_Y = \{ U \cap Y : U \in \beta \}$$

is a basis for the subspace topology on Y.

Example

The order topology on $[0, 1]$ is the subspace topology on $[0, 1]$ viewed a subspace of $\mathbf{R}$.