Topology

Definition

Let X be a set. Then a topology, τ, on X is a collection of subsets of X such that

1. $\emptyset, X \in \tau$.
2. If $U_1, \ldots, U_n \in \tau$, then $\bigcap_{j=1}^{n} U_j \in \tau$.
3. If $U_i \in \tau$ for all $i \in I$, then $\bigcup_{i \in I} U_i \in \tau$.

Remark

Item (2) just says τ is closed under finite intersections. Item (3), says that τ is closed under arbitrary unions.

Definition

A topological space is a pair (X, τ) consisting of a set X and a topology τ in X.

Motivating Example

Example

Let $X = \mathbf{R}^n$ and let τ be the collection of open sets in $\mathbf{R}^n$. Then $(\mathbf{R}^n, \tau)$ is a topological space. We call τ the usual topology on $\mathbf{R}^n$.

Example

Note that a subset $U \subset \mathbf{R}$ is open exactly when every point in U is contained in an interval contained in U.

Remark

Because of these examples, in any topological space (X, τ), elements of τ are called open sets.

Generic Examples

Example (The Discrete Topology)

If X is any set, then the power set $\mathcal{P}(X)$ is a topology on X called the discrete topology.

Remark

Note that (X, τ) has the discrete topology if and only if $\{x\}$ is open for all $x \in X$.

Example (The Trivial Topology)

If X is any set, then $\tau = \{\emptyset, X\}$ is a topology on X called the trivial topology.