Math 56 Compu & Expt Math, Spring 2013: Topics Weeks 4-6 (Midterm 2)

1 Week 4

Definition of $\|f\|$, 2-norm of a function on an interval, definition of orthogonality.

Definition of complex Fourier series on $(0, 2\pi)$. Modes $\{e^{-imx}\}$ are orthogonal. Projection formula to extract coefficients given f. Real-valued functions have conjugate symmetric coefficients.

Parseval's relation $\|f\|^2 = 2\pi \sum_{m\in\mathbb{Z}} |\hat{f}_m|^2$.

Best approximation property (in L^2-norm on $(0, 2\pi)$) of coefficients $\hat{f}_m$ for a truncated Fourier series.

DFT for signals length N: formula for $\tilde{f}_m$ for $m = 0, \ldots, N-1$. DFT as a matrix F (and its later formula in terms of ω), or linear map from $\mathbb{C}^N$ to itself.

Derivation of DFT from N-node equispaced quadrature approx to projection formula. f_j are "normalized samples" $\frac{1}{N} f(2\pi j/N)$

2 Week 5

Magnitude of complex number z is $\sqrt{z^* z}$. Conjugate transpose of vector or matrix. Generalization of 2-norm to complex vectors $\|\mathbf{f}\| = \sqrt{\mathbf{f}^*\mathbf{f}}$.

Sum lemma for powers of root of unity $\omega = e^{2\pi i/N}$.

Inversion formula for DFT (flip sign of power of ω, overall $1/N$ factor)

Unitarity of $\frac{1}{\sqrt{N}}F$, equivalence to inversion formula. Consequences: length preservation under action of $\frac{1}{\sqrt{N}}F$, i.e. discrete Parseval's theorem.

Aliasing: DFT coeffs in terms of true Fourier coeffs: $\tilde{f}_m = \cdots + \hat{f}_{m-N} + \hat{f}_m + \hat{f}_{m+N} + \cdots$ Interpretation that DFT coeffs from 0 to $N/2 - 1$ approximate same Fourier coeffs, while DFT coeffs $N/2 + 1$ to $N - 1$ approximate Fourier coeffs $-N/2 + 1$ to -1.

Nyquist sampling theorem: $N/2$-band-limited function exactly reconstructed from N samples, by its Fourier series using the DFT coefficients. "Two points per period."

Cooley–Tukey FFT. Basic idea (don't need to memorize twiddles and formulae), its complexity.

3 Week 6

Trigonometric polynomial interpolation = $N/2$-truncated Fourier series with coefficients estimated via DFT of N samples. Error in max norm $\leq 2\sum_{|n|>=N/2} |\hat{f}_n|$.

Smoother f have faster-decaying coefficients. $f \in C^k$ means $\hat{f}_m = O(|m|^{-k})$. Definition, examples of, and how to test for super-algebraic convergence.

Interpreting Fourier index m as physical frequency m/T where T is total sample time.

Convolution theorem: $\widehat{f*g} = \hat{f}\hat{g}$. Application to audio and image processing. Basic deconvolution and the issue of noise amplification if certain $\hat{g}_m$ coeffs close to zero.

Periodic (cyclic) vs acyclic convolution, zero-padding so that the former computes the latter.

Basics of arbitrary-precision (integer only) operations in some base b.

Strassen's fast multiply using convolution via FFT, complexity vs schoolbook multiply.

Newton iteration for reciprocal, what the point of it is.

Euler–Machin formulae for π (not the obscure ones, but how to derive the one with 1/2 and 1/3), convergence rate of Taylor series of $\tan^{-1}$. Complexity is at least $O(N^2)$ for N digits.

Existence of Brent–Salamin iteration for π (not the whole formula!). You should know: what the arithmetic-geometric mean iteration is, quadratic convergence.

4 Practise questions

1. Compute the sum of squared magnitudes of the Fourier coefficients of the periodized function $f(x) = 1 + x$ on $(0, 2\pi)$.
2. Prove that if $f(x)$ is real-valued, then its Fourier coeffs obey the symmetry $\hat{f}_n = \hat{f}^*_{-n}$.
3. Find the acyclic convolution of [1 2 3 4] and [12]. What length arrays are needed to compute it periodically?
4. Prove that a function with Fourier series $\hat{f}_n = O(|n|^{-5})$ has a best N-term trigonometric polynomial interpolant whose max error converges at least as fast as $O(N^{-4})$.
5. What is the best approximation to the periodic function $f(x) = x$ on $(0, 2\pi)$ in the 2-norm sense, over all functions of the form $ae^{-ix} + b + ce^{ix}$?
6. Write down the $N = 2$ DFT matrix. Explain where it appears in the FFT algorithm.
7. Show how the Fourier series coefficients of f' relate to those of f.
8. Find an unbounded function f that has finite 2-norm on $(0, 2\pi)$ [tricky]
9. if you sampled $f(x) = e^{imx}$ for $m = 17$ on a regular 2π-periodic grid of size $N = 10$, what would the DFT of this sample vertor be?
10. Consider $f(x) = \sin(7x) + \cos(9x)$. How many samples on a regular grid in $[0, 2\pi)$ are sufficient to reconstruct f exactly from its Fourier series?
11. Derive the Newton iteration for reciprocal and explain why important.
12. Prove that $F^*F = NI$ where F is the DFT matrix. What is consequence for inverting the DFT?

13. Give as many theorems/results as you can whose proof relies on the sum lemma for the Nth principal root of unity, and state each one.

5 Some practise question answers

1. Use Parseval! (don't compute the coeffs directly)
2. Use projection formula.
3. [1 4 7 10 8] I think. $N \geq 5$.
4. In lecture 12 5/7/13 we proved the max trig interp error is $\leq 2\sum_{|n|\geq N/2} |\hat{f}_n|$. Bounding the sum from $N/2$ upwards by the integral, as in the algebraic tails of lecture 1, we have error $O(N^{-4})$.
5. a, b, c are just the Fourier coeffs of the given f, byt the best approximation property (which you proved, using orthogonality).
6. $[11; 1 - 1]$. Appears at the lowest recursion level in FFT.
7. mult by im. From taking deriv inside integral.
8. $f(x) = 1/x^\gamma$ for any $0 < \gamma < 1/2$ will do. The spike is still in L^2.
9. Aliasing formula says $\tilde{f}_7 = 1$ and $\tilde{f}_j = 0$ for $j \neq 7$. Ie the vector $[0\,0\,0\,0\,0\,0\,0\,1\,0\,0]^T$.
10. Nyquist says $N = 20$ sufficient. (Actually $N = 19$ enough but we've tended to assume N even. I'd grade either correct.)
11. See lecture. Important since enables division to be done using just a few multiplies, which are fast.
12. Write using ω and use sum lemma. $F^{-1} = (1/N)F^*$ which is nearly same as DFT itself.
13. Inversion formula for DFT, unitarity of $\frac{1}{\sqrt{N}}F$, F^2 is N times a permutation matrix, aliasing formula, Nyquist sampling theorem, convolution theorem.