Math 56 Compu & Expt Math, Spring 2013: Midterm 1

4/18/13, pencil and paper, 2 hrs, 50 points. Good luck!

1. [9 points]

   (a) Say a computer's algorithm for e^x has relative error in the output of up to $\varepsilon_{\text{mach}}$, for $-1 \leq x \leq 1$. Does this guarantee that the algorithm is *backward stable* in this domain?

   (b) Repeat the question for $\sin x$ in the same domain.

   (c) For some x outside $[-1, 1]$ one of the above algorithms cannot be backward stable. Which one, and for what x?

2. [8 points] Consider $f(x) = 1/(2 + x^2)$.

(a) What type, and order/rate, do you expect for convergence of the Taylor series truncated to terms less than x^n, expanding about the origin, when evaluated at $x = 0.5$? Explain

Write an upper bound on the error reflecting this convergence, in big-O notation:

(b) Estimate up to what power x^n is needed for this series to reach 16-digit accuracy.

3. [8 points] Consider the "left-sided" finite-difference approximation $f'(x) \approx \frac{f(x)-f(x-h)}{h}$

(a) Derive a rigorous bound on the error that applies to each $h > 0$ [Hint: your bound will need to involve properties of f]

(b) What axes would one choose on a graph so that the error appears as a straight line and yet data at $h = 10^{-4}, 10^{-8}, 10^{-12}$ are all visible?

(c) Explain what happens to the error of the approximation in practice as $h \to 0$

BONUS Roughly what h has the smallest error?

4. [7 points] Consider the linear system $\begin{bmatrix} 1 & 0 \\ 10^5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

How many digits accuracy (relative to the solution norm $\sqrt{x_1^2 + x_2^2}$) are you guaranteed in the solution if the system is solved by a backward stable algorithm with $\varepsilon_{\text{mach}} = 10^{-16}$?

[You may assume a constant of 1 in the backward stability. Hint: full points for rigorous upper bound on the error; generous partial credit for intelligent estimates or other bounds]

[BONUS] Find a right-hand side **b** for which the above worst-case prediction is (nearby) achieved.

5. [7 points] Given $y > 0$, you wish to approximate $x = \sqrt{y}$ using elementary operations.

(a) Derive a Newton iteration that converges to the desired x [Hint: x must be a root of something]

(b) Derive a big-O estimate on the error ε_n after n iterations.

6. [11 points] Short answers.

(a) Prove whether $N + \dfrac{N}{(\log_{10} N) - 7} = O(N)$, giving, if true, a constant and corresponding condition on N.

(b) Prove whether $\sqrt{1+x^2}\sin x = o(x)$ as $x \to \infty$

(c) How close to 1 does x have to be such that the relative condition number of computing $\sqrt{x-1}$ is 10^8?

(d) Prove that $\|A^{-1}\| = \left(\min_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}\right)^{-1}$

(e) Give a definition of an algorithm $\tilde{f}(x)$ for a problem $f(x)$ being *stable*: