Barnett 4/11/13

SOLUTIONS

Math 56 Compu & Expt Math, Spring 2013: Quiz 1

in class 4/11/13, 25 mins, just pencil and paper

[4] 1. Prove whether $\underbrace{\frac{\cos(x)}{x-100}}_{f(x)} = O(\underbrace{1/x}_{g(x)})$ as $x \to \infty$

need $\left|\frac{f(x)}{g(x)}\right| \le C$ for $\forall x > x_0$

$$\left|\frac{f(x)}{g(x)}\right| = \left|\frac{x\cos x}{x-100}\right| = \left|\frac{\cos x}{1-\frac{100}{x}}\right|$$

but $1 - \frac{100}{x} > 1/2$ for all $x > 200$

$\le 2 \quad \forall x > 200 \quad \Rightarrow$ proved, yes.

[4] 2. Estimate, giving working, the *relative error* in computing $1000.001 - 1000$ with a machine using standard "double precision" arithmetic.

this integer is exactly stored (but let's assume it's rounded like any other number)

$y = 10^{-3}$ true answer.

$$\hat{y} = fl(1000.001) \ominus fl(1000)$$

$$= [1000.001\,(1+\varepsilon_1) - 1000\,(1+\varepsilon_2)]\,(1+\varepsilon_3)$$

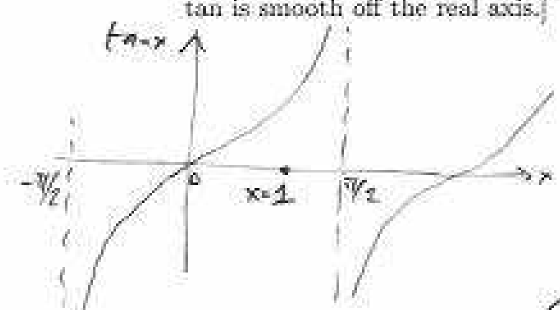
officially there, but can ignore since much smaller.

$$\approx 10^{-3} + \underbrace{1000.001\,\varepsilon_1 + 1000\,\varepsilon_2}$$

abs err worst case $\approx 2 \cdot 10^3 \varepsilon_{mach}$

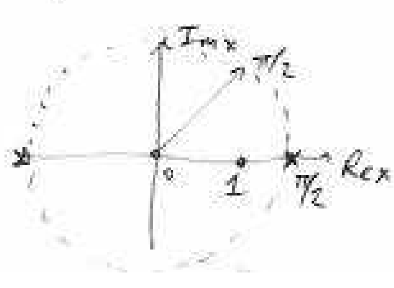
$\Rightarrow$ rel. err $\frac{|\hat{y}-y|}{|y|} \le \frac{2\cdot 10^3}{10^{-3}}\varepsilon_{mach} \approx 2 \times 10^{-10}$, or 10^{-10} if used $\varepsilon_2 = 0$.

[4] 3. We wish to approximate $\tan x$ at $x = 1$ by the n-term Taylor series expanding about the origin. What type, and order/rate, of convergence would you expect? Explain. [Hint: you don't need the series, and tan is smooth off the real axis.]



tan has singularities (poles) at $x = \pm\pi/2$ which are the nearest to the origin (since told tan smooth off the real axis)

Taylor series converge out to radius $\pi/2$.



Convergence (by thm in class) is exponential with rate $r = \frac{\text{dist from } x \text{ to center}}{\text{dist from singularity to center}} = \frac{1}{\pi/2} = \frac{2}{\pi} < 1$

$\varepsilon = O(r^n)$ is another way to say.

[5] 4. We wish to approximate $\sin x$ at $x = 0.1$ by the first non-trivial term in its Taylor series expanding about the origin. Give a rigorous bound on the error.

Back to Taylor's thm:

remainder term

$$\sin x = \underbrace{0}_{f(0)} + x \underbrace{\cos(0)}_{f'(0)} + \underbrace{(-\sin(q))}_{f''(q)} \frac{x^2}{2!}$$

1st nontrivial term is just x.

for some $q \in [0, x]$

$$\text{so } |\sin x - x| \le \underbrace{|\sin q|}_{\text{can bound by 1 rigorously}} \frac{x^2}{2}$$

(could do better) eg. $|\sin q| \le q$

$$\le x^2/2$$

so abs. error $|\sin 0.1 - 0.1| \le \frac{10^{-2}}{2} = 0.005$

If you used $|\sin q| < q$ you got $\frac{x^3}{2} = 0.0005$.

[2] 5. What is the relative condition number κ of computing $\underbrace{1/(x-1)}_{f(x)}$?

$$\kappa = \kappa(x) = \left|\frac{x f'(x)}{f(x)}\right| = \left|\frac{x \cdot \frac{-1}{(x-1)^2}}{1/(x-1)}\right| = \left|\frac{x}{x-1}\right|$$

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It is possible to do even better, if you write the remainder term at x^3, ie:

$$\sin x = x + f'''(q)\frac{x^3}{3!} \quad \text{for some } q \in (0, x]$$

$$\text{so } |\sin x - x| \le |\cos q| \cdot \frac{x^3}{6} \le \frac{x^3}{6} \approx 0.00017$$

I didn't expect this.