Math 56 Compu & Expt Math, Spring 2014: Midterm 2

5/13/14, pencil and paper, 2 hrs, 50 points. Show working. Good luck!

1. [7 points]

   (a) Let $\mathbf{f} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ be a vector in $\mathbb{C}^2$. What is $\|\mathbf{f}\|$, its 2-norm ?

   (b) Considering the above $\mathbf{f}$, what is $\|\tilde{\mathbf{f}}\|$, where $\tilde{\mathbf{f}}$ is the DFT of $\mathbf{f}$?

   (c) Let $\mathbf{f}$ and $\mathbf{g}$ be general vectors in $\mathbb{C}^N$. Say their inner product $\mathbf{f}^*\mathbf{g} = 0$, then what can you say about $\tilde{\mathbf{f}}^*\tilde{\mathbf{g}}$? Prove it.

2. [8 points]

(a) Compare computing a Fourier series and a discrete Fourier transform, by stating on what objects they *act* and what they *produce*.

(b) State the DFT formula, taking f_j to $\tilde{f}_m$, and its inverse DFT formula to go the other way:

(c) Prove the inverse DFT formula works:

BONUS. What is the effect on a vector f of taking the DFT, then complex conjugating $\tilde{f}$, then inverse DFT?

3. [10 points] Consider the 2π-periodic function $f(x) = \cos 3x$.

(a) Compute $\|f\|$ using the usual $L_2([0, 2\pi))$ norm. [Hint: there is more than one way to do this]

(b) If this $f(x)$ is sampled on a uniform grid with $N = 4$ points, give the resulting DFT vector $\tilde{f}$, and the reconstructed interpolant $\sum_{|m|<N/2} \tilde{f}_m e^{imx}$ that results. [Hint: if stuck, sketch]

(c) What range of N results in this f being interpolated exactly from its N-point sampled DFT coefficients?

(d) For this f, what is the set of three coefficients c_m, $m = -1, 0, 1$, that minimizes the L_2 norm of the error $\sum_{|m|<2} c_m e^{imx} - f(x)$?

4. [9 points]

(a) Compute the acyclic convolution of [1 2 3] with [-1 2].

(b) Explain how to most efficiently compute the acyclic convolution of two long vectors of length N and M, stating i) any theorem used, ii) the working vector length needed, and iii) the overall complexity.

(c) An unknown image f has been convolved by a known aperture function g to give a measured signal h. Explain precisely why recovering f from h often leads to a very "noisy" answer.

5. [6 points] Consider the function

$$f(x) = \begin{cases} 1, & 0 \le x < \pi/2, \\ 0, & \pi/2 \le x < 2\pi. \end{cases}$$

(a) Compute the Fourier series coefficients $\hat{f}_n$, making sure your formula covers all n:

(b) Explain how they are consistent with a theorem relating decay of Fourier coefficients to smoothness of f.

BONUS. Explain how to generate the complete Fourier series for the antiderivative of the deviation of a general function from its average value.

6. [10 points] Short-answer questions.

(a) Say a 2π-periodic function f has Fourier coefficients $\hat{f}_n$ with third-order algebraic decay in $|n|$. What can you prove about the convergence rate of N-point trigonometric polynomial interpolation, as $N \to \infty$?

(b) Estimate the number of *iterations* of Brent–Salamin's algorithm needed to get π accurate to a billion (10^9) digits.

(c) Estimate the number of Taylor series terms (expanding $\tan^{-1}$ about the origin) needed to get π correct to a million digits using $\pi/4 = 2\tan^{-1} 1/3 + \tan^{-1} 1/7$.

(d) Prove whether $e^{-\sqrt{n}}$ has super-algebraic decay to zero as $n \to \infty$.

(e) What is the complexity of computing the square-root of a number to N digit accuracy, for N large? (You may assume constant effort per flop in the FFT).