Barnett
5/13/14.

~ SOLUTIONS ~

Math 56 Compu & Expt Math, Spring 2014: Midterm 2

5/13/14, pencil and paper, 2 hrs, 50 points. Show working. Good luck!

1. [7 points]

2. (a) Let $\mathbf{f} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ be a vector in $\mathbb{C}^2$. What is $\|\mathbf{f}\|$, its 2-norm ?

$$\|f\| = \sqrt{f^* f} = \sqrt{[1-i, 1]\begin{bmatrix} 1+i \\ 1 \end{bmatrix}} = \sqrt{|1+i|^2 + |1|^2}$$

$$= \sqrt{2+1} = \sqrt{3}$$

2. (b) Considering the above $\mathbf{f}$, what is $\|\tilde{\mathbf{f}}\|$, where $\tilde{\mathbf{f}}$ is the DFT of $\mathbf{f}$?

Two ways: i) Discrete Parseval $\|\tilde{f}\| = \sqrt{N}\,\|f\| = \sqrt{2}\cdot\sqrt{3} = \sqrt{6}$

or ii) $\tilde{f} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{F^{(2)}\ \text{DFT matrix.}} \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} 2+i \\ i \end{bmatrix}$ so $\|\tilde{f}\| = \sqrt{|2+i|^2 + |i|^2}$

$$= \sqrt{5+1} = \sqrt{6}.$$

3, (c) Let $\mathbf{f}$ and $\mathbf{g}$ be general vectors in $\mathbb{C}^N$. Say their inner product $\mathbf{f}^*\mathbf{g} = 0$, then what can you say about $\tilde{\mathbf{f}}^*\tilde{\mathbf{g}}$? Prove it.

so $\vec{f}$ & $\vec{g}$ orthogonal.

Since DFT is rotation (w/ scaling by $\sqrt{N}$) in $\mathbb{C}^N$, angles preserved

$\Rightarrow$ $\tilde{f}$ & $\tilde{g}$ orthogonal too.

Proof: $\tilde{f}^*\tilde{g} = (Ff)^*(Fg) = f^* \underbrace{F^* F}_{NI} g$ (identity, $N\times N$)

$$= N f^* g = 0 \text{ if } f^* g = 0.$$

2. [8 points]

2. (a) Compare computing a Fourier series and a discrete Fourier transform, by stating on what objects they *act* and what they *produce*.

Fourier series (ie computing the series given $f(x)$) acts on 2π-periodic function f to give coeffs. $\{\hat{f}_n\}_{n\in\mathbb{Z}}$

DFT acts on signal vector $f \in \mathbb{C}^N$ to give a coeff vector $\tilde{f} \in \mathbb{C}^N$

3. (b) State the DFT formula, taking f_j to $\tilde{f}_m$, and its inverse DFT formula to go the other way:

$$\tilde{f}_m = \sum_{j=0}^{N-1} \omega^{-mj} f_j \qquad f_j = \frac{1}{N}\sum_{m=0}^{N-1} \omega^{jm} \tilde{f}_m \qquad \omega := e^{\frac{2\pi i}{N}}$$

3. (c) Prove the inverse DFT formula works: subst for $\tilde{f}_m$ into inverse formula (written w/ k for j)

For any $k=0,\dots,N-1$,
$$f_k = \frac{1}{N}\sum_{m=0}^{N-1} \omega^{km}\tilde{f}_m = \frac{1}{N}\sum_{m=0}^{N-1}\omega^{km}\sum_{j=0}^{N-1}\omega^{-mj} f_j \qquad (*)$$

swap sums, as always!

$$= \sum_{j=0}^{N-1} f_j \frac{1}{N}\sum_{m=0}^{N-1}\omega^{(k-j)m}$$

by Sum lemma $= \begin{cases} N & k=j \bmod N \\ 0 & \text{otherwise.}\end{cases}$

$$= \sum_{j=0}^{N-1} f_j \frac{1}{N}\cdot N\delta_{kj}$$

but $j,k \in \{0,\dots,N-1\}$ so $k=j$ is only nonzero case.

$$= f_k$$

recovered signal component f_k, as hoped, for general f.

BONUS. What is the effect on a vector f of taking the DFT, then complex conjugating $\tilde{f}$, then inverse DFT?

Go to $(*)$ above but with conjugate of $\tilde{f}_m$, so $\frac{1}{N}\sum_{m=0}^{N-1}\omega^{km}\sum_{j=0}^{N-1}\omega^{+mj} f_j^*$

So we get $\sum_{j=0}^{N-1} f_j^* \delta_{k+j \bmod N, 0} = f^*_{N-j} = \begin{bmatrix} f_0^* \\ f_{N-1}^* \\ \vdots \\ f_1^* \end{bmatrix}$ } reversed order.

3. [10 points] Consider the 2π-periodic function $f(x) = \cos 3x$.

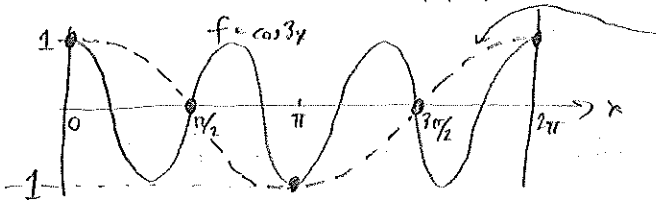
2 (a) Compute $\|f\|$ using the usual $L_2([0, 2\pi))$ norm. [Hint: there is more than one way to do this]

i) $\|f\| = \sqrt{\int_0^{2\pi} |f(x)|^2 dx}$ by defn. $= \sqrt{\int_0^{2\pi} \cos^2 3x \, dx} = \sqrt{\int_0^{2\pi} \left[\frac{1}{2} + \underbrace{\tfrac{1}{2}\cos 6x}_{\text{integral zero}}\right] dx}$

$= \sqrt{2\pi \cdot \frac{1}{2}} = \sqrt{\pi}$

ii) Parseval: nonzero Fourier coeffs of $\cos 3x$ are $\hat{f}_3 = \hat{f}_{-3} = 1/2$, $\|f\|^2 = 2\pi \sum |\hat{f}_n|^2 = \pi$ agrees.

4 (b) If this $f(x)$ is sampled on a uniform grid with $N = 4$ points, give the resulting DFT vector $\tilde{f}$, and the reconstructed interpolant $\sum_{|m|<N/2} \tilde{f}_m e^{imx}$ that results. [Hint: if stuck, sketch]



interpolant, is $\cos x$, graphically.
Let's prove it.
Fourier coeffs are $\hat{f}_3 = \hat{f}_{-3} = 1/2$, others zero.
since $\cos 3x = \dfrac{e^{3ix} + e^{-3ix}}{2}$

So aliasing gives $\tilde{f}_m = \cdots \hat{f}_{m-4} + \hat{f}_m + \hat{f}_{m+4} + \hat{f}_{m+8} + \cdots$

$$\tilde{f} = \begin{array}{c} m=0 \\ 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1/2 \end{bmatrix} \begin{array}{l} \\ \leftarrow \text{from } \hat{f}_{1-4} \\ \\ \leftarrow \text{from } \hat{f}_{3+0} \end{array}$$

$$\sum_{|m|<2} \tilde{f}_m e^{imx} = \frac{1}{2}e^{-ix} + \frac{1}{2}e^{ix} = \cos x$$

matches sketch

2 (c) What range of N results in this f being interpolated exactly from its N-point sampled DFT coefficients?

Nyquist says $N >$ twice the highest freq. in the function
> 6

2 (d) For this f, what is the set of three coefficients c_m, $m = -1, 0, 1$, that minimizes the L_2 norm of the error $\sum_{|m|<2} c_m e^{imx} - f(x)$?

The $\{c_m\}$ that minimize the error in L_2-norm are precisely the true Fourier coeffs. (of Fourier series) for f, namely $c_m = \hat{f}_m$

Thus $c_{-1} = \hat{f}_{-1} = 0$, $c_0 = \hat{f}_0 = 0$, $c_1 = \hat{f}_1 = 0$ All zero.

(perhaps surprisingly!)

Note, they are <u>not</u> the DFT coeffs.

indices: i=0,1,2   i=0,1

4. [9 points]

2. (a) Compute the acyclic convolution of [1 2 3] with [−1 2].

f   g

$$\begin{array}{r} -1\ -2\ -3 \\ 2\ \ 4\ \ 6 \\ \hline [-1\ \ 0\ \ 1\ \ 6] \end{array}$$

or via $(f *_A g)_j = \sum_{i \in \mathbb{Z}} f_i g_{j-i}$

with zero assumed when index i falls outside its allowed range.

output index: $j = 0, 1, 2, 3$

5. (b) Explain how to most efficiently compute the (acyclic) convolution of two long vectors of length N and M, stating i) any theorem used, ii) the working vector length needed, and iii) the overall complexity.

Zero-pad vectors each to length $\underbrace{n}_{\text{working vector length}} = N + M - 1$:

$[\overbrace{f_0 \cdots f_{N-1}}^{N}\ 0\ 0\ 0]$
$[\underbrace{g_0 \cdots g_{M-1}}_{M}\ 0\ 0\ 0\ 0]$
$\longleftrightarrow n.$

Then use Convolution Theorem: $\widetilde{(f * g)} = \tilde{f}\tilde{g}$

Let f, g be the zero-padded vectors length n.

Use FFT to get $\tilde{f}, \tilde{g}$, cost $O(n \log n)$

this has nothing to do with Strassen's fast mult!

compute $\tilde{h}_m = \tilde{f}_m \tilde{g}_m$, $m = 0, \ldots, n-1$ cost $O(n)$

Take inverse FFT to get $h = F^{-1}\tilde{h} = f * g$ by theorem.
at cost $O(n \log n)$

Overall complexity: $O(n \log n)$

2. (c) An unknown image f has been convolved by a known aperture function g to give a measured signal h. Explain precisely why recovering f from h often leads to a very "noisy" answer.

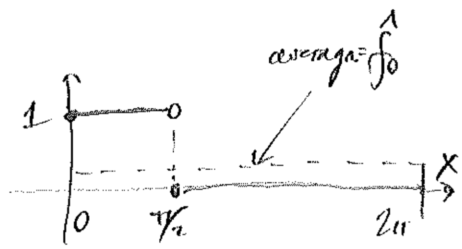
Deconvolution done by:

$h \xrightarrow{\text{FFT}} \tilde{h}_m$
$g \xrightarrow{\text{FFT}} \tilde{g}_m$

pointwise divide $\tilde{f}_m = \dfrac{\tilde{h}_m}{\tilde{g}_m}$

$f \xleftarrow{(\text{FFT})^{-1}} \tilde{f}_m$

If h, and thus $\tilde{h}_m$, contains measurement noise, and any of the $\tilde{g}_m$ are v. small, this noise amplified by $1/\tilde{g}_m$ & has a massive effect on reconstructed $\tilde{f}_m$, hence f.



5. [6 points] Consider the function

$$f(x) = \begin{cases} 1, & 0 \le x < \pi/2, \\ 0, & \pi/2 \le x < 2\pi. \end{cases}$$

4. (a) Compute the Fourier series coefficients $\hat{f}_n$, making sure your formula covers all n:

$$\hat{f}_n = \frac{1}{2\pi}(e^{inx}, f) = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi/2} e^{-inx} \cdot 1\,dx$$

Case $n = 0$ (easiest) : $\hat{f}_0 = \frac{1}{2\pi}\int_0^{\pi/2} 1\,dx = 1/4$ average value.

$n \neq 0$: $\hat{f}_n = \frac{1}{2\pi}\left[\frac{1}{-in} e^{-inx}\right]_0^{\pi/2} = \frac{i}{2\pi n}\left(e^{-in\pi/2} - 1\right)$

$= \frac{i}{2\pi n}\left((-i)^n - 1\right)$ will do.

2. (b) Explain how they are consistent with a theorem relating decay of Fourier coefficients to smoothness of f.

f has zero bounded derivatives, so theorem (from HW) says $\hat{f}_n = o(1)$ ($k=0$) Above we observe $\hat{f}_n = O\left(\frac{1}{|n|}\right)$

ie, merely decay to zero. which is consistent with theorem.

BONUS. Explain how to generate the complete Fourier series for the antiderivative of the deviation of a general function from its average value.

Opposite of taking derivative $\hat{f}_n \to in\hat{f}_n$.

call $g = f - \hat{f}_0$ then g has Fourier series $\begin{cases} \hat{g}_n = \hat{f}_n, & n \neq 0 \\ \hat{g}_0 = 0. \end{cases}$

Let $h(x) = \int_0^x g(y)\,dy + C$ be antiderivative, ie $h'(x) = g(x)$

then $h'(x) = \sum_{n\in\mathbb{Z}} in\hat{h}_n e^{inx} = \sum_{n\in\mathbb{Z}} \hat{g}_n e^{inx}$ & by orthog., we equate each term,

$\Rightarrow \hat{h}_n = \frac{\hat{g}_n}{in} = \frac{\hat{f}_n}{in}$ for $n \neq 0$, while $\hat{h}_0$ is arbitrary.

6. [10 points] Short-answer questions. hat not tilde.

1. (a) Say a 2π-periodic function f has Fourier coefficients $\hat{f}_n$ with third-order algebraic decay in $|n|$. What can you prove about the convergence rate of N-point trigonometric polynomial interpolation, as $N \to \infty$?

$\hat{f}_n = O\left(\frac{1}{|n|^3}\right)$ N-pt. Trig. poly interpolation max error E_N

$$E_N \leq 2\sum_{|n|\geq N/2} |\hat{f}_n| \leq 4\sum_{n\geq N/2} \frac{C}{n^3} \leq C\int_{N/2}^{\infty} \frac{dn}{n^3} = CN^{-2}$$

Error $= O\left(\frac{1}{N^2}\right)$

2. (b) Estimate the number of *iterations* of Brent–Salamin's algorithm needed to get π accurate to a billion (10^9) digits.

Quadratically convergent, so doubles # correct digits each iteration; assume start from 1 correct.

iters

$$10^9 = 2^n \quad \text{so} \quad n = \frac{\log 10^9}{\log 2} = 9\,\frac{\log 10}{\log 2} \approx 30$$

only!

2. (c) Estimate the number of Taylor series terms (expanding $\tan^{-1}$ about the origin) needed to get π correct to a million digits using $\pi/4 = 2\tan^{-1} 1/3 + \tan^{-1} 1/7$.

the larger x, dominates with rate $r = 1/3$.

Taylor series error $\approx$ error in last term (up to constant, when exponentially conv.)

Need $r^n \approx 10^{-\text{million}}$ ie. $n = \frac{\log 10^{-10^6}}{\log r} = -10^6 \frac{\log 10}{\log 1/3} = 10^6 \frac{\log 10}{\log 3}$

$\approx 2.2 \times 10^6$

2. (d) Prove whether $e^{-\sqrt{n}}$ has super-algebraic decay to zero as $n \to \infty$.

means $= O(n^{-k})$, for each $k > 0$

General k:

$$\frac{f(n)}{g(n)} = \frac{e^{-\sqrt{n}}}{n^{-k}} = \frac{n^k}{e^{\sqrt{n}}} \overset{\text{l'Hôp}}{\to} \frac{kn^{k-1}}{\frac{1}{2}n^{-1/2}e^{\sqrt{n}}} = 2k\,\frac{n^{k-1/2}}{e^{\sqrt{n}}} \overset{\text{l'Hôp}}{\to} \cdots$$

can repeat, lowering power of n by $1/2$ each time.

$\cdots \to C_k \frac{1}{e^{\sqrt{n}}} \to 0$ as $n \to \infty$ $\Rightarrow$ yes.

2. (e) What is the complexity of computing the square-root of a number to N digit accuracy, for N large? (You may assume constant effort per flop in the FFT).

Sqrt done by Newton iterations, takes $O(\ln N)$ iterations.

Each such iteration requires reciprocal, done via $O(\ln N)$ Newton iters, each using Strassen's fast multiply via FFT convolution, $O(N \ln N)$.

Total is $O(N \log^3 N)$.