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5/7/14.

~ SOLUTIONS ~

Math 56 Compu & Expt Math, Spring 2014: Quiz 2

in X-hr 5/7/14, 35 mins, just pencil and paper

3. 1. (a) Compute the periodic convolution of $\overset{f}{[1,1,2]}$ with $\overset{g}{[1,0,-1]}$. $N=3$

$$(f*g)_j = \sum_{i=0}^{2} f_i \, g_{j-i \,(\mathrm{mod}\ 3)} \qquad j=0,1,2$$

Or do by adding shifted copies:

$$\begin{array}{rll} & 1\cdot[1\ 1\ 2] & \text{unshifted } f \\ & +\,0\cdot[2\ 1\ 1] & f \text{ shifted 1 right} \\ & -\,1\cdot[1\ 2\ 1] & \text{" } \ \text{2 right.} \\ = & [0,-1,1] & \end{array}$$

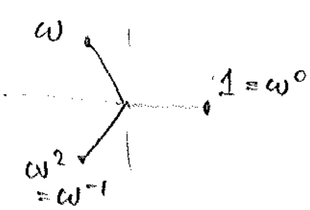
2. (b) To what length should these vectors be zero-padded so that their periodic convolution correctly computes the acyclic one?

$$\underbrace{\begin{array}{ccccc} [1 & 1 & 2] & & \\ & & [1 & 0 & -1] \end{array}}_{\text{acyclic. max length } = 5}$$

$$N_1 + N_2 - 1 = 3+3-1 = 5.$$

3. 2. Write down the middle row of the $N=3$ DFT matrix (if you use symbols, define them).

Let $\omega = e^{\frac{2\pi i}{N}} = e^{\frac{2\pi i}{3}}$



$$\begin{array}{lccc} & j=0 & j=1 & j=2 \\ \text{Row } m=1 \text{ is} & [\,\omega^0 & \omega^{-1} & \omega^{-2}\,] \end{array}$$

$$= \left[1 \;\; e^{\frac{4\pi i}{3}} \;\; e^{\frac{2\pi i}{3}}\right] \qquad \text{since } F_{mj} = \omega^{-mj}$$

3\. 3. (a) Say the function $f(x) = e^{-3ix}$ is sampled on a regular grid of size $N = 8$. What DFT coefficient vector $\tilde{f}$ would result?

$$\hat{f}_n = \begin{cases} 1 & n = -3 \\ 0 & \text{otherwise} \end{cases} \quad \text{is Fourier series for } f. \text{ (unique)}$$

Sampling & DFT gives $\tilde{f}_m = \cdots + \boxed{\hat{f}_{m-N}} + \hat{f}_m + \hat{f}_{m+N} + \cdots$

Usual indices of DFT output are $0 \le m < N$, so $m = +5$ gives $\tilde{f}_5 = \hat{f}_{5-8} = \hat{f}_{-3} = 1$.

$$\tilde{f}_m = \begin{array}{cccccccc} m=0 & & & & & m=5 & & \\ \end{array}$$

$$\tilde{f}_m = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0] \quad \text{or its transpose.}$$

$\uparrow m = 0$. $\quad \uparrow m = 5$.

2\. (b) What function results when trigonometric polynomial interpolation on this same grid is used to reconstruct f from the DFT coefficients you just computed? Comment.

Trig. poly interp. $\sum_{|n|<N} \tilde{f}_n e^{inx}$ with $\tilde{f}_n$ N-periodic as usual.

$$= \underbrace{\tilde{f}_{-3}}_{=\tilde{f}_5} e^{i(-3)x} \quad \text{since all others zero.}$$

$[0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0]$

freq 0, 1, 2, 3, -3, -2, freq -1

$$= e^{-3ix}$$

This is identical to input; freq. $3 < \frac{N}{2} = 4$, the Nyquist frequency.

3\. 4. A function has Fourier series decaying as $\hat{f}_n = O(1/|n|^3)$. It is sampled on a regular N-point grid and the DFT taken. Prove an optimal big-O bound on the decay vs N of $|\tilde{f}_0 - \hat{f}_0|$, ie the error in the approximated average value.

Aliasing formula. $\tilde{f}_m = \cdots \hat{f}_{m-2N} + \hat{f}_{m-N} + \hat{f}_m + \hat{f}_{m+N} + \hat{f}_{m+2N} + \cdots$

Choose $m = 0$, so

$$|\tilde{f}_0 - \hat{f}_0| = |\hat{f}_N + \hat{f}_{2N} + \cdots + \hat{f}_{-N} + \hat{f}_{-2N} + \cdots|$$

from big-O on Fourier series

$$\le \frac{C}{N^3} + \frac{C}{(2N)^3} + \cdots + \frac{C}{N^3} + \frac{C}{(2N)^3} + \cdots$$

$$= \frac{2C}{N^3} \underbrace{\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots\right)}_{\text{some const.}} = O\left(\frac{1}{N^3}\right)$$

Same order of convergence as the series decays at.

sum of, tail of series.

[BONUS: prove a big-O bound on the interpolation error]

From lecture, interp. error $E_N \le 2\sum_{|n|\ge N} |\hat{f}_n| \le 4C\sum_{n\ge N} \frac{1}{n^3}$ integral test.

$$\le 4C\int_N^\infty n^{-3}\,dn = \frac{C}{N^2} = O\left(\frac{1}{N^2}\right)$$

One order worse than Fourier series decay.

4. 5. Let f and g be length-N signal vectors. What is $\tilde{f} * \tilde{g}$ (periodic convolution) in terms of f and g? Prove it. [Hint: this is a kind of inverted convolution theorem.]

[This was hard]

$$\tilde{f}_m = \sum_{j=0}^{N-1} \omega^{-mj} f_j \quad , \quad \tilde{g}_n = \sum_{i=0}^{N-1} \omega^{-ni} g_i$$

Note I used new indices for later.

So $(\tilde{f}*\tilde{g})_n = \sum_{m=0}^{N-1} \tilde{f}_m \tilde{g}_{n-m \pmod N}$ by defn. of periodic convolution

$$= \sum_{m=0}^{N-1} \sum_{j=0}^{N-1} \omega^{-mj} f_j \sum_{i=0}^{N-1} \omega^{-(n-m \bmod N)i} g_i$$

can drop since $\omega^N = 1$.

usual trick: drag inner sums out w/ as much stuff as goes with them.

$$= \sum_j \sum_i f_j g_i \sum_m \omega^{-ni} \omega^{m(i-j)}$$

also const., & use sum lemma. $\sum_m \omega^{m(i-j)} = \begin{cases} N & i=j \pmod N \\ 0 & \text{otherwise} \end{cases} = \delta_{ij}$

$$= \sum_j \sum_i f_j g_i \omega^{-ni} \cdot N\delta_{ij}$$

turns $i \to j$ & kills one sum.

$$= N \sum_j \omega^{-nj} f_j g_j$$

$$= N(\widetilde{fg})_n$$

$\leftarrow n^{th}$ component of DFT of fg (pointwise product)

Compare usual convolution thm $\widetilde{f*g} = \tilde{f}\tilde{g}$