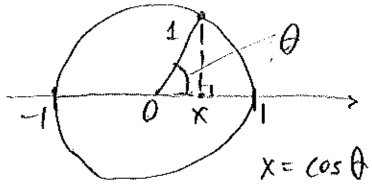
Barnett 5/21/13

MATH 56 WORKSHEET: Clenshaw Curtis Quadrature

$$I = \int_0^{2\pi} f(\theta) g(\theta)\, d\theta \overset{\text{Parseval}}{=} 2\pi \sum_{m \in \mathbb{Z}} \hat{f}_m \hat{g}_m^* \qquad (P)$$



Assume: i) $\hat{f}_m \approx \tilde{f}_m = \sum_{j=0}^{N-1} \omega^{-mj} \frac{f(\theta_j)}{N}$

ii) truncate Fourier sum to $\sum_{|m|<N/2}$ as in interpolation

A) Substitute i) & ii) into (P) & rearrange to get $I = \sum_{j=0}^{N-1} f(\theta_j) \cdot (\text{something})$:

B) The "something" must be the elements of a weight vector $\vec{w} = \{w_j\}_{j=0}^{N-1}$

Assuming $\hat{g}_m$ are known, give the fastest scheme to fill $\vec{w}$:

C) Nearly 1/2 the evaluations of f are wasted! — why?

For N even, write a smaller set of weights for nodes 0 to N/2 only:

D) Evaluate Fourier coeffs $\hat{g}_m$ for $g(\theta) = \frac{1}{2}|\sin\theta|$ use the back!

~ SOLUTIONS ~

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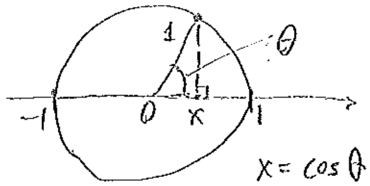


Assume: i) $\hat f_m \approx \tilde f_m = \sum_{j=0}^{N-1} \omega^{-mj} \frac{F(\theta_j)}{N}$

ii) truncate Fourier sum to $\sum_{|m|<N/2}$ as in interpolation

A) Substitute i) & ii) into (P) & rearrange to get $I = \sum_{j=0}^{N-1} F(\theta_j)\cdot(\text{something})$:

$$I = 2\pi \sum_{|m|<N/2} \sum_{j=0}^{N-1} \omega^{-mj} \frac{1}{N} F(\theta_j)\, \hat g_m \overset{\text{swap sums}}{=} \sum_{j=0}^{N-1} F(\theta_j) \cdot \underbrace{\frac{2\pi}{N} \sum_{|m|<N/2} \omega^{-mj} \hat g_m}_{w_j}$$

Note since $\hat g_m = \hat g_{-m}$ (follows since $g(\theta)$ even & real), can write

$$w_j = \frac{2\pi}{N} \sum_{|m|<N/2} \omega^{jm} \hat g_m = 2\pi\, \mathrm{DFT}^{-1} \{\hat g_m\}_{m=-N/2+1}^{m=N/2-1}$$

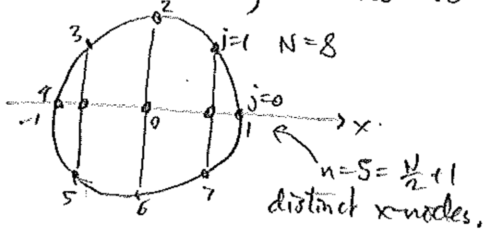
B) The "something" must be the elements of a weight vector $\vec w = \{w_j\}_{j=0}^{N-1}$

Assuming $\hat g_m$ are known, give the fastest scheme to fill $\vec w$: use fast Fourier transform ↓

fill $\vec g = [\hat g_0, \hat g_1, \dots, \hat g_{N/2-1}, 0, \hat g_{-N/2+1}, \hat g_{-N/2+2}, \dots, \hat g_{-1}]$, then $\vec w = 2\pi\, \mathrm{FFT}^{-1}(\vec g)$

C) Nearly ½ the evaluations of f are wasted! — why? since some nodes $x_j = \cos\theta_j$ overlap $x_j = x_{N-j}$.

For N even, write a smaller set of weights for nodes 0 to $N/2$ only:



so $I \approx \sum_{j=0}^{N/2} w'_j\, f(\cos \frac{2\pi j}{N})$ with $\vec w' = [w_0, 2w_1, 2w_2, \dots, 2w_{N/2-1}, w_{N/2}]$

D) Evaluate Fourier coeffs $\hat g_m$ for $g(\theta) = \frac{1}{2}|\sin\theta|$ use the back! $\hat g_m = \begin{cases} \frac{1}{\pi}\frac{1}{m^2-1} & m \text{ even}, \\ 0 & m \text{ odd} \end{cases}$