Barnett
8/2/13

MATH 56 WORKSHEET : Euler–Machin formulae for π

Start w/ addition formula for tangent: $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

A) Set $\alpha = \tan x$, $\beta = \tan y$ & $\tan^{-1}$ both sides:

B) Check $\alpha = 1/2$, $\beta = 1/3$ works (Euler)

If want to push $\alpha \to 0$, what happens to β if result is to be $\tan^{-1} 1$?

C) We can further split $\tan^{-1} 1/2 = \tan^{-1} 1/3 + \tan^{-1}(\text{something})$

Solve for "something":

Thus write $\pi/4 = \ldots$

Why is this preferred?

D) to get π to N digits how many Taylor terms needed? (big-O)

So, what is total complexity? (big-O)

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SOLUTIONS

Start w/ addition formula for tangent: $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

A) Set $\alpha = \tan x$, $\beta = \tan y$ & $\tan^{-1}$ both sides: $x = \tan^{-1}\alpha$, $y = \tan^{-1}\beta$

$$\tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}\frac{\alpha+\beta}{1-\alpha\beta}$$

B) Check $\alpha = 1/2$, $\beta = 1/3$ works (Euler) $\dfrac{\alpha+\beta}{1-\alpha\beta} = \dfrac{1/2 + 1/3}{1 - 1/6} = \dfrac{5/6}{5/6} = 1$

so $\tan^{-1} 1/2 + \tan^{-1} 1/3 = \tan^{-1} 1$.

If want to push $\alpha \to 0$, what happens to β if result is to be $\tan^{-1} 1$?

Fix $\dfrac{\alpha+\beta}{1-\alpha\beta} = 1$, solve for β: $\beta = \dfrac{1-\alpha}{1+\alpha}$ so as $\alpha \to 0^+$, $\beta \to 1^-$

C) We can further split $\tan^{-1} 1/2 = \tan^{-1} 1/3 + \tan^{-1}(\text{something})$

Solve for "something": $\dfrac{\alpha+\beta}{1-\alpha\beta} = 1/2$ so $\beta = \dfrac{1/2 - \alpha}{1 + \alpha/2} = \dfrac{1/2 - 1/3}{1 + 1/6} = \dfrac{1/6}{7/6}$
$= 1/7$

Thus write $\pi/4 = 2\tan^{-1} 1/3 + \tan^{-1} 1/7 = 2\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \cdots\right]$
$+ \frac{1}{1\cdot 7} - \frac{1}{3\cdot 7^3} + \frac{1}{5\cdot 7^5} - \cdots$

Why is this preferred? convergence rate is largest $|x| = 1/3 = r$, ie error $= O\left(\frac{1}{3^n}\right)$, beats $r = 1/2$ above

D) to get π to N digits how many Taylor terms needed? (big-O)

$\hookrightarrow \left(\frac{1}{3}\right)^2 \approx \frac{1}{10}$ so roughly $2N = O(N)$ terms.

So, what is total complexity? (big-O)

$O(N^2)$, assuming $O(n)$ to update $\frac{1}{3^n} \to \frac{1}{3^{n+1}}$ each term