~~~ SOLUTIONS ~~~

Math 56 Compu & Expt Math, Spring 2014: Quiz 1

in class 4/10/14, 25 mins, just pencil + paper + brain

[4] 1. Prove whether $\underbrace{10^3+n}_{f(n)} = O(\underbrace{n}_{g(n)})$ as $n \to \infty$ (if so, give C and n_0)

$$\left|\frac{f(n)}{g(n)}\right| = \frac{10^3+n}{n} = \frac{10^3}{n}+1 \le 2$$

for all $n > 10^3$

So, yes is big-O with $C = 2$ & $n_0 = 10^3$

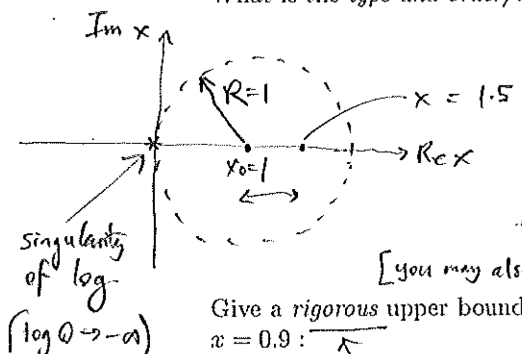
[$C = 1001$ & $n_0 = 1$ also possible, but worse constant]

2. The Taylor expansion of log about $a = 1$ is

[3] ie, x_0

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

What is the *type and order/rate* of convergence of this series when evaluated at $x = 1.5$?



The nearest singularity of log is at the origin (see lecture), a distance of $R=1$ from $x_0 = 1$.

Exponential convergence with rate $r = \dfrac{|x-x_0|}{R} = \dfrac{0.5}{1} = \frac{1}{2}$

[you may also do by bounding tail of series].

Give a *rigorous* upper bound on the absolute error in approximating $\log x$ by $\underbrace{(x-1)-(x-1)^2/2}_{\text{terms up to } n=2 \text{ of Taylor series, @ } x_0=1}$ at $x = 0.9$:

[4] means we need Taylor's theorem:

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \frac{(x-x_0)^3}{3!}f'''(q)$$

for some $q \in [x, x_0]$

$$\text{so} \quad \log x = 0 + \underbrace{(x-1) - \frac{(x-1)^2}{2}}_{\text{our approximation}} + \frac{(x-1)^3}{3!}\underbrace{\frac{d^3}{dx^3}\log x\Big|_{x=q}}_{\frac{1}{2q^3}}$$

error term.

$$\text{So, } |\text{error}| = \left|\frac{(x-1)^3}{3q^3}\right| = \frac{0.1^3}{3|q^3|} \le \frac{0.1^3}{3(0.9)^3} = \frac{1}{3000(0.9)^3}$$

worst-case q is 0.9
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[5] 3. Estimate, giving working, the *relative error* in computing 100.00001 - 100 with a machine using standard "double precision" arithmetic.

Let's use fact that 100 represented exactly.

$$\text{machine ans} = \Big(\underbrace{100.00001\,(1+\varepsilon_1)}_{\text{rounding input}} - 100\Big)\underbrace{(1+\varepsilon_2)}_{\text{due to } \ominus\text{, machine subtraction}}$$

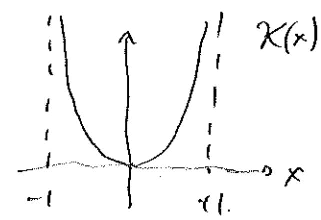
much smaller than dominant one.

$$= \underbrace{10^{-5}}_{\text{exact ans.}} + \underbrace{\varepsilon_1\, 100.00001}_{\text{dominant error } \lesssim 10^2 \varepsilon_{mach}} + 10^{-5}\varepsilon_2 + \cancel{O(\varepsilon^2)}$$

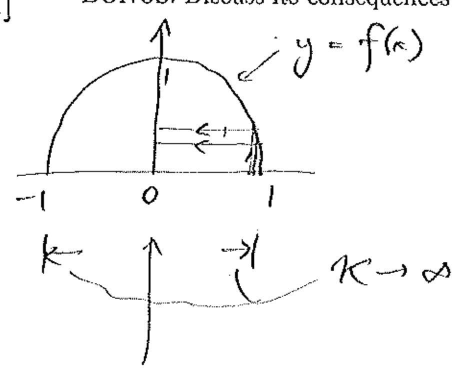
$$\text{Relative error} = \frac{|y-\hat{y}|}{|y|} \le \frac{10^2\varepsilon_{mach}}{10^{-5}} \approx 10^{+7}\varepsilon_{mach} \approx 10^{-9}$$

[4] 4. What is the relative condition number $\kappa(x)$ of the function $f(x) = \sqrt{1-x^2}$ in $-1 \le x \le 1$?

$$\kappa(x) = \left|\frac{f'(x)\,x}{f(x)}\right| = \left|\frac{-2x\frac{1}{2}(1-x^2)^{-1/2}\,x}{(1-x^2)^{1/2}}\right| = \frac{x^2}{1-x^2}$$



[0-2] BONUS: Discuss its consequences for the machine evaluation error of this function over the interval.



function is semicircle.

- $\kappa \to \infty$ at $x \to \pm 1$

  so we cannot expect machine evaluation to be relatively accurate there, even if $\tilde{f}$ is backward stable alg. for f.

- $\kappa = 0$ at $x = 0$, so expect highest accuracy there. However, we cannot expect relative error to get smaller than ε_{mach} even though $\kappa \to 0$. This means we have to give up backwards stability in favor of merely stability.

Advanced:

$$\left|\frac{\tilde{f}(x) - f(\tilde{x})}{f(\tilde{x})}\right| = O(\varepsilon_{mach})$$

for some $\tilde{x} = x(1+\varepsilon)$.