# Proof of superexponential convergence

Math 56

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Consider the Taylor series for exp about $x = 0$,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Here we prove that this series is *super-exponentially* convergent, i.e. for each rate $r > 0$, no matter how small, $\varepsilon$, the error in truncating to $n$ terms, is $O(r^n)$.

This implies that on a semi-log plot ($\log \varepsilon$ vs $n$), the graph has increasingly negative slope.

*Proof.* Given location $x$, and rate $r > 0$, then choose integer $N > |x|/r$. Then, for $k \geq N$, a single term has the bound

$$\frac{|x^k|}{k!} \leq \frac{N^N}{N!} \cdot \frac{|x|^k}{N^k} \leq Cr^k$$

(Why? Make sure you understand this. $C$ can be large but is always const wrt $k$.) So the tail is bounded as usual by $\varepsilon \leq C \sum_{k>n} r^k = O(r^n)$ for all $n \geq N$. □

Note that this arbitrarily high rate of exponential convergence corresponds to $e^x$ being analytic in arbitrarily large discs around the origin. This is called an *entire* function.