Math 56 Compu & Expt Math, Spring 2014: Topics Weeks 1,2

1 Week 1

Relative vs absolute error

Big O, little o. Know definitions, be able to test if one func is O or o of another, as some parameter goes large or small.

Algebraic convergence, order. Know how to bound tail of algebraic series by integral.

Exponential convergence, rate. How to bound tail by pulling out a geometric series. Thm that rate is asymptotically (dist from eval pt to center)/(nearest dist of singularity to center)

How to choose good axes for a plot so data spread and linear, interpret slope.

Definition of superexponential convergence.

Basic complex arithmetic, magnitude-phase notation.

2 Week 2

Taylor's theorem with correct remainder term, using it to bound err. eg using to prove super-exp conv for exp(x)

Newton's iteration. Definition of quadratic convergence (ie $\varepsilon_{n+1}/\varepsilon_n^2 \leq C$), sketch of proof that Newton's is quadr conv. Newton's for computing sqrt. Bisection alg from HW.

Set of floating point numbers, their gaps, error due to rounding ie $f\ell(x)$. Defn of $\varepsilon_{\text{mach}}$. Rules of floating point arithmetic (rounding combined with $+ - \times /$)

Sum numbers in magnitude smallest to largest, and why other order is worse.

Catastrophic cancellation, spotting it and predicting its size by following epsilons; using math to rewrite a formula to avoid it.

Relative condition number of a problem $\kappa(x)$, defn, consequence for expectation of relative accuracy for evaluation of a function.

3 Practise questions

Also see worksheets, homeworks, and Quiz 1 from 2013. More to follow.

1. Is $\frac{e^n}{10-ne^n} = O(n^{-1})$ as $n \to \infty$? Prove it.

2. Prove if $\log x = O(x)$ as $x \to \infty$? As $x \to 0$?

3. Is $\log n = o(\log(n^2))$ as $n \to \infty$? If so prove it; if not, what else can be said?

4. is $\cos(n)e^{-\sqrt{n}} = O(n^{-10})$ as $n \to \infty$?

5. Write a Newton iteration to solve $x^3 - x = 1$. What function of the error creates a linear graph when plotted vs iteration number n?

6. Use Taylor's theorem to give a simple upper bound on the absolute error in approximating $\cos x$ by $1 - x^2/2$ which applies in $|x| < 0.5$.

7. Fixing any $x > 0$ and $r > 0$, show that the n-term Taylor series for e^x about 0 has error $O(r^n)$. State the type of convergence this implies.

8. Estimate the relative error introduced when a floating point machine evaluates $f(x) = 1 + x$.

9. Write all solutions to $z^3 = 8i$ in the form $re^{i\theta}$.

4 Some practise question answers

1. Yes. To prove use $10/(ne^n) < 2$ for all $n > 1$, so $n_0 = 1$ and $C = 2$.
2. l'Hôpital's rule both times. Ans: Yes, no.
3. No. But it's big-O.
4. Yes, and it's little-o too.
5. $x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$. Quadratic convergence, so $\log(\log 1/\varepsilon_n)$ vs n linear.
6. We're using terms 0, 1 (which has zero coefficient), and 2 here, so the thm says error (call ε) is bounded by the next (3rd) term but with x in the derivative replaced by unknown q in the interval. $\varepsilon = f'''(q)(x-0)^3/3!$. So an upper bound over the interval is $|\varepsilon| \leq (.5)^3/3! = 1/48$.
7. Taylor theorem, then $\lim_{n\to\infty}(x/r)^n/n! = 0$, so is bounded by a const for all sufficiently large n. Super-exponential convergence.
8. No more than $\frac{2|x|+1}{x+1}\varepsilon_{\text{mach}}$.
9. geom gives $2e^{i\theta}$ where $\theta = \pi/6, 5\pi/6, 9\pi/6$ since each angle when tripled gives $\pi/2$